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Some results on determinants and inverses of nonsingular pentadiagonal matrices

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ABSTRACT

A block matrix analysis is proposed to justify, and modify, a known algorithm for computing in $O(n)$ time the determinant of a nonsingular $n \times n$ pentadiagonal matrix ($n \geq 6$) having nonzero entries on its second subdiagonal. Also, we describe a procedure for computing the inverse matrix with acceptable accuracy in $O(n^2)$ time. In the general nonsingular case, for $n \geq 5$, proper decompositions of the pentadiagonal matrix, as a product of two structured matrices, allow us to obtain both the determinant and the inverse matrix by exploiting low rank structures.

1. Introduction

A nonsingular $n \times n$ matrix $\mathbf{P} = \{p_{ij}\}_{1\leq i,j\leq n}$ is pentadiagonal if $p_{i,j} = 0$ for $|i - j| > 2$. These play an important role in contemporary numerical analysis. They arise frequently in numerical methods for solving ordinary and partial differential equations, interpolation schemes, and spline problems, [1]. Also, pentadiagonal matrices appear in fine approximations of second order derivatives, and in boundary value problems involving fourth order derivatives. Gaussian methods with partial pivoting are usually used for the inversion of such matrices. However, these methods can destroy the low rank structure and sparsity of pentadiagonal matrices; e.g. by row-interchange operations. Therefore, specialized techniques adapted to the low rank structure of pentadiagonal matrices are of interest.

Some specific parallel and sequential algorithms for the inversion of pentadiagonal matrices are already known. A recursive procedure for calculating in $O(n^2)$ time the inverse matrix $\mathbf{P}^{-1}$ of a pentadiagonal matrix $\mathbf{P}$ having nonzero entries on its second superdiagonal, $p_{i,j} \neq 0$ for $j - i = 2$, was given in [2]. In [3] there was proposed a different sequential procedure, having computational complexity $O(n^2)$, for pentadiagonal matrices having an *LU* (Doolittle) factorization.

Fast numerical algorithms for computing the determinants of pentadiagonal matrices are also needed to test efficiently for the existence of unique solutions of partial differential equations, and for solving the inverse problem of constructing

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symmetric pentadiagonal Toeplitz matrices. Some methods having complexity $O(n)$ have been obtained; see e.g. [4–8]. Building upon such results, in Section 2 we introduce a block matrix analysis to justify, in terms of matrix cofactors, the algorithm given in [6] for computing with complexity $O(n)$ the determinant, $\det \mathbf{P}$, of a pentadiagonal matrix having nonzero entries on its second subdiagonal. This kind of matrix is currently used in numerical methods. Since it is not hard to do, we find it convenient subsequently to adapt this algorithm to compute in $O(n^2)$ time the entire inverse matrix $\mathbf{P}^{-1}$, up to an acceptable accuracy. Analogous results can also be obtained for pentadiagonal matrices with nonzero entries on their second superdiagonals.

A specific procedure for computing both the determinant and the inverse of any nonsingular pentadiagonal matrix $\mathbf{P}$, taking advantage of its low rank structure, with no further conditions on its entries, remains an open question. In Section 3 we propose factorizations appropriate for the general nonsingular case where the pentadiagonal matrix $\mathbf{P}$ is decomposable as a product of two structured matrices; e.g. upper Hessenberg matrices (see also [9]). This enables us to exploit the low rank structure of (sparse) structured matrices, including triangular, tridiagonal, and Hessenberg matrices, to compute both the determinant, $\det \mathbf{P}$, and the inverse $\mathbf{P}^{-1}$. Illustrative comparisons, examples, and remarks are also presented.

2. Pentadiagonal matrices having nonzero entries on their second subdiagonals

For an $n \times n$ ($n \geq 6$) nonsingular pentadiagonal matrix with nonzero entries on its second subdiagonal we assume the 2×2 block structure,

$$\mathbf{P} = \left(\begin{array}{c|c} \mathbf{P}_{11} & \mathbf{0}_2 \\ \hline \mathbf{U} & \mathbf{P}_{22} \end{array}\right). \tag{1}$$

The submatrices $\mathbf{P}_{11}$ and $\mathbf{P}_{22}$ have dimensions $2 \times n-2$ and $n-2 \times 2$, respectively. The matrix $\mathbf{0}_2$ is the 2×2 zero matrix. The $n-2 \times n-2$ nonsingular matrix $\mathbf{U}$ is upper triangular. The transposed partition,

$$\mathbf{P}^{-1} = \left(\begin{array}{c|c} -\mathbf{U}^{-1}\mathbf{P}_{22}\mathbf{M}_{21} & \mathbf{U}^{-1} + \mathbf{U}^{-1}\mathbf{P}_{22}\mathbf{M}_{21}\mathbf{P}_{11}\mathbf{U}^{-1} \\ \hline \mathbf{M}_{21} & -\mathbf{M}_{21}\mathbf{P}_{11}\mathbf{U}^{-1} \end{array}\right), \tag{2}$$

of its inverse is well known; see e.g. [10]. Here, $\mathbf{M}_{21} = \frac{1}{\det \mathbf{P}} \begin{pmatrix} C_{1,n-1} & C_{2,n-1} \\ C_{1,n} & C_{2,n} \end{pmatrix}$ is calculated using the classical Cayley cofactor formula for the inverse. The $C_{i,j}$ are cofactors of $\mathbf{P}$. Therefore, the inverse matrix

$$\mathbf{P}^{-1} = \begin{pmatrix} -\mathbf{U}^{-1}\mathbf{P}_{22} \\ \mathbf{I}_2 \end{pmatrix} \mathbf{M}_{21} \begin{pmatrix} \mathbf{I}_2 & -\mathbf{P}_{11}\mathbf{U}^{-1} \end{pmatrix} + \left(\begin{array}{c|c} \mathbf{0}_{n,2} & \mathbf{U}^{-1} \\ \hline \mathbf{0}_2 & \mathbf{0}_{2,n} \end{array}\right), \tag{3}$$

can be seen as a rank two perturbation of a strictly upper triangular matrix, [10]. All the information required for the inversion of $\mathbf{P}$ is contained in the submatrices $\mathbf{M}_{21}$ and $\mathbf{U}^{-1}$. As a result, we can calculate $\mathbf{P}^{-1}$ using simple matrix products as in (3).

2.1. Computing the determinant in $O(n)$ time

A compact expression for calculating the determinant of a nonsingular pentadiagonal matrix having nonzero entries on its second superdiagonal was given in [6]. It also applies to a matrix having nonzero entries on its second subdiagonal. A sequential algorithm for computing $\det \mathbf{P}$ with complexity $O(n)$ was also given. In order to justify, in terms of a computation using matrix cofactors, the formula for $\det \mathbf{P}$ given in [6], we introduce a second pentadiagonal matrix, $\mathbf{P}^*$, associated with $\mathbf{P}$ and having ones on its second subdiagonal. A variant of this related algorithm, together with (3), allows us to compute the full inverse matrix $\mathbf{P}^{-1}$.

Proposition 1. *Let $\mathbf{P}$ be an $n \times n$ ($n \geq 6$) nonsingular pentadiagonal matrix having nonzero entries on its second subdiagonal. With $\mathbf{P}$ we associate the matrix $\mathbf{P}^* = \mathbf{P} \cdot \mathbf{diag}\left(\frac{1}{p_{31}}, \frac{1}{p_{42}}, \ldots, \frac{1}{p_{n,n-2}}, 1, 1\right)$. The determinant of $\mathbf{P}$ is given by*

$$\det \mathbf{P} = \left(\prod_{k=1}^{n-2} p_{k+2,k}\right) \det \begin{pmatrix} C^*_{1,n-1} & C^*_{2,n-1} \\ C^*_{1n} & C^*_{2n} \end{pmatrix}, \tag{4}$$

where the C^*_{ji} are cofactors of the matrix $\mathbf{P}^*$. Moreover, $\det \mathbf{P}$ can be computed in $O(n)$ time.

Proof. First, we note that $\det \mathbf{P} = \left(\prod_{k=1}^{n-2} p_{k+2,k}\right) \det \mathbf{P}^*$. Then we must demonstrate that $\det \mathbf{P}^* = \det \begin{pmatrix} C^*_{1,n-1} & C^*_{2,n-1} \\ C^*_{1n} & C^*_{2n} \end{pmatrix}$. The matrix $\mathbf{P}^*$ is pentadiagonal, with ones on its second subdiagonal.

Partitioning $\mathbf{P}^*$ as in (1) and $\mathbf{P}^{*-1}$ as in (2), we obtain a partition of the identity matrix $\mathbf{I}_n$, where $\left(\mathbf{P}^*\mathbf{P}^{*-1}\right)_{11} = \mathbf{I}_2$. That is, $-\mathbf{P}^*_{11}\mathbf{U}^{*-1}\mathbf{P}_{22}\mathbf{M}^*_{21} = \mathbf{I}_2$. Since the matrix $\mathbf{U}^*$ in (1) is upper triangular with ones on its main diagonal, applying the nullity theorem [11], we conclude that the 2×2 matrix entry $\mathbf{M}^*_{21}$, in the transposed partition of $\mathbf{P}^{*-1}$, is nonsingular. Therefore, we have

$$\frac{1}{\det \mathbf{P}^*} \begin{pmatrix} C^*_{1,n-1} & C^*_{2,n-1} \\ C^*_{1n} & C^*_{2n} \end{pmatrix} = \mathbf{M}^*_{21} = \left(-\mathbf{P}^*_{11}\mathbf{U}^{*-1}\mathbf{P}_{22}\right)^{-1}. \tag{5}$$