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A general spline differential quadrature method based on quasi-interpolation

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ABSTRACT

The differential quadrature method is a numerical discretization technique for the approximation of derivatives. The classical method is polynomial-based, and there is a natural restriction in the number of grid points involved. A general spline-based method is proposed to avoid this problem. For any degree a Lagrangian spline interpolant is defined having a fundamental function with small support. A quasi-interpolant is used to achieve the optimal approximation order. That two-stage scheme is detailed for the cubic, quartic, quintic and sextic cases and compared with another methods that appear in the literature.

1. Introduction

The differential quadrature method (DQM) is a numerical discretization technique for the approximation of derivatives by means of weighted sums of function values. It was proposed by Bellman and coworkers in the early 1970s, and it has been extensively employed to approximate spatial partial derivatives (cf. [1,2] and references quoted therein). The classical DQM is polynomial-based, and it is well known that the number of grid points involved is usually restricted to be below 30. Some spline based DQMs have been proposed to avoid this problem. Given a *B*-spline (cf. [3,4]), a cardinal Lagrangian or hermitian spline with compactly supported fundamental function is defined, from which the approximation of the derivatives is derived. But the construction of this spline interpolant depends strongly on the degree of the *B*-spline (see for instance [5–7]).

In this work we present a general DQM based on interpolation and quasi-interpolation. We are interested in constructing compactly supported spline functions L satisfying the interpolation conditions

$$L(j) = \delta_{j,0}, \quad j \in \mathbb{Z}. \tag{1}$$

Here, δ stands for Kronecker's delta, i.e.

$$\delta_{j,0} := \begin{cases} 1, & \text{if } j = 0, \\ 0, & \text{if } j \neq 0. \end{cases}$$

This is achieved in a two-stage process: firstly, a general method is given to define such a kind of fundamental function having a small support, but the corresponding interpolation operator $\mathcal{L}$ does not reproduce the polynomials included in the space spanned by the integer translates of L; then, a new interpolation operator that reproduces those polynomials is defined by using the Boolean sum of $\mathcal{L}$ with an appropriate quasi-interpolation operator.

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In Section 2 the Fourier transform is used to characterize the functions L satisfying (1). After fixing a particular structure for L as a linear combination of functions derived from a continuous function ϕ with compact support, such a function L is also characterized in terms of the existence of common zeros in $\mathbb{C} \setminus \{0\}$ of two Laurent polynomials associated with ϕ. In Section 3, we present a general method to define fundamental functions with compact support and the results when the B-spline centered at the origin is considered, as well as the explicit expressions of those fundamental functions for B-splines of low degree. In Section 4 classical quasi-interpolation operators are considered to produce interpolation operators achieving the optimal approximation order, and the cases associated with B-splines of low order are fully developed. In Section 5, some results due to Zhong and collaborators are revised and compared with the corresponding schemes defined in this paper.

2. Characterizing the fundamental functions

The construction of spline functions L, compactly supported or not, satisfying the interpolation conditions (1) has been widely studied (see, e.g. [8–13]). The periodization of the Fourier transform

$$\widehat{L}(u) := \int_{\mathbb{R}} \exp\left(-iux\right) L(x) dx$$

of L is a useful method to characterize the fulfillment of interpolation conditions (1) (cf. [12, Theorem 2]).

Proposition 1. *Let L be a exponentially decaying function and $\widehat{L}$ be the Fourier transform of L. Then L fulfills the interpolation conditions (1) if and only if $\sum_{j\in\mathbb{Z}} \widehat{L}\left(u + 2\pi j\right) = 1$.*

We will suppose that L can be written as

$$L = \sum_{j\in J} c_j \phi\left(2\cdot -j\right), \tag{2}$$

J being a finite subset of $\mathbb{Z}$ and ϕ a compactly supported continuous function. The next result will permit to establish the relationship between the periodizations of L and ϕ.

Lemma 2. *The following equalities hold:*

$$\sum_{j\in\mathbb{Z}} \widehat{\phi}\left(\frac{u+2\pi j}{2}\right) = 2\sum_{j\in\mathbb{Z}} e^{-\mathrm{i}ju}\phi(2j),$$

$$\sum_{j\in\mathbb{Z}} (-1)^j \widehat{\phi}\left(\frac{u+2\pi j}{2}\right) = 2\sum_{j\in\mathbb{Z}} e^{-i\left(j+\frac{1}{2}\right)u}\phi\left(2j+1\right).$$

Proof. For $k \in \{0, 1\}$, let $f(x) := e^{-ixu}\phi\left(2x+k\right)$. The change of variable $x \to 2x + k$ provides the equality

$$\widehat{\phi}\left(\frac{u+2\pi j}{2}\right) = 2\int_{\mathbb{R}} e^{-\mathrm{i}(2x+k)\frac{u+2\pi j}{2}}\phi\left(2x+k\right) dx.$$

Then,

$$\widehat{\phi}\left(\frac{u+2\pi j}{2}\right) = 2\left(-1\right)^{jk} e^{-ik\frac{u}{2}} \int_{\mathbb{R}} e^{-i(2\pi j)x} f(x)dx,$$

and so

$$\widehat{\phi}\left(\frac{u+2\pi j}{2}\right) = 2\left(-1\right)^{jk} e^{-ik\frac{u}{2}}\widehat{f}\left(2\pi j\right).$$

Applying the Poisson summation formula $\sum_{j\in\mathbb{Z}}\widehat{f}\left(2\pi j\right) = \sum_{j\in\mathbb{Z}} f(j)$, we get

$$\sum_{j\in\mathbb{Z}} (-1)^{jk}\widehat{\phi}\left(\frac{u+2\pi j}{2}\right) = 2e^{-ik\frac{u}{2}}\sum_{j\in\mathbb{Z}} e^{-\mathrm{i}ju}\phi\left(2j+k\right),$$

and the proof is complete. □

The existence of compactly supported functions satisfying (1) is then characterized by Proposition 1 and Lemma 2 (see [9, p. 288]).

Proposition 3. *Let ϕ be a compactly supported continuous function. There exists a fundamental function of the form (2) for some finitely supported $c := \left(c_j\right)_{j\in\mathbb{Z}}$ of finite support if and only if the Laurent polynomials*

$$\Phi_0(z) := \sum_{j\in\mathbb{Z}} \phi(2j)z^{2j} \quad \textit{and} \quad \Phi_1(z) := \sum_{j\in\mathbb{Z}} \phi\left(2j+1\right) z^{2j+1} \tag{3}$$