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# Low-complexity root-finding iteration functions with no derivatives of any order of convergence[☆]

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ABSTRACT

In this paper, a procedure to design Steffensen-type methods of different orders for solving nonlinear equations is suggested. By using a particular divided difference of first order we can transform many iterative methods into derivative-free iterative schemes, holding the order of convergence of the departure original method. Numerical examples and the study of the dynamics are made to show the performance of the presented schemes and to compare them with another ones.



## 1. Introduction

Solving nonlinear equations is a classical problem which has interesting applications in various branches of science and engineering. In this study, we describe new iterative methods to find a simple root $\alpha$ of a nonlinear equation $f(x) = 0$, where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a scalar function on an open interval $I$.

In the last years, a lot of papers have developed the idea of removing derivatives from the iteration function in order to avoid defining new functions, and calculate iterates only by using the function that describes the problem, trying to preserve the convergence order. The interest of these methods is to be applied on nonlinear equations when there are many problems for obtaining and evaluating the derivatives involved, or when there is no analytical function to derive.

The known Newton's method for finding $\alpha$ uses the iterative expression

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, \ldots$$

which converges quadratically in some neighborhood of $\alpha$. If the derivative $f'(x_k)$ is replaced by the forward-difference approximation

$$f'(x_k) \approx f[z_k, x_k] = \frac{f(z_k) - f(x_k)}{z_k - x_k}, \tag{1}$$

where $z_k = x_k + f(x_k)$, Newton's method becomes

$$x_{k+1} = x_k - \frac{f(x_k)}{f[z_k, x_k]}, \quad k = 0, 1, \ldots$$

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which is the known Steffensen's method, (see [1]). This scheme is a touch competitor of Newton's method. Both are of second order, both require two functional evaluations per step, but in contrast to Newton's method, Steffensen's scheme is derivative-free.

Commonly, the efficiency of an iterative method is measured by the efficiency index defined by Ostrowski in [2] as $p^{1/d}$, where $p$ is the order of convergence and $d$ is the number of functional evaluations per step. Kung and Traub conjectured in [3] that the order of convergence of any multipoint method cannot exceed the bound $2^{d-1}$. The schemes that reach this bound are called optimal methods.

In this paper, by using the idea of Steffensen's scheme, we are going to design a procedure that allows to transform many iterative schemes for solving nonlinear equations into derivative-free method, preserving the order of convergence.

The rest of the paper is organized as follows: in Section 2 we describe the mentioned procedure, design some optimal derivative-free iterative schemes and establish the convergence order of these methods. In Section 3 different numerical tests, using smooth and non-smooth functions, and functions with zeros of multiplicity greater than one, confirm the theoretical results and allow us to compare the new methods with the starting ones. In Section 4, some dynamical aspects associated to the presented methods are studied. We finish this manuscript with some conclusions and remarks.

## 2. Development of the procedure

Traub in [4] presented an iterative method with third order of convergence, which needs three functional evaluations per step, two of the function $f$ and one of the derivative $f'$. It is known that if we replace $f'$ by the forward-difference approximation, the resulting method has the iterative expression

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f[z_k, x_k]}, \\ x_{k+1} &= y_k - \frac{f(y_k)}{f[z_k, x_k]}, \end{aligned}$$

where $z_k = x_k + f(x_k)$, and it has order of convergence three. So, we ask the following question:

Every time that $f'$ is replaced by the forward-difference approximation, do you always preserve the order of convergence?

The answer is negative as we can see now. It is known that Ostrowski's method, given by the iterative expression

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= y_k - \frac{f(x_k)}{f(x_k) - 2f(y_k)} \frac{f(y_k)}{f'(x_k)}, \end{aligned}$$

has order of convergence four and uses three functional evaluations per iteration, so it is an optimal method in the sense of Kung–Traub conjecture. However, if we use the mentioned approximation of $f'$, the resulting scheme

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f[z_k, x_k]}, \\ x_{k+1} &= y_k - \frac{f(x_k)}{f(x_k) - 2f(y_k)} \frac{f(y_k)}{f[z_k, x_k]}, \end{aligned} \tag{2}$$

where $z_k = x_k + f(x_k)$, has only order of convergence three, being its error equation

$$e_{k+1} = -c_2^2(1 + f'(\alpha)) \left[3 + f'(\alpha) + 2(1 + f'(\alpha))^2\right] e_k^3 + O(e_k^4),$$

where $c_j = \frac{f^{(j)}(\alpha)}{j! f'(\alpha)}$ for $j = 2, 3, \ldots$ and $e_k = x_k - \alpha$.

Nevertheless, if we use $z_k = x_k + f(x_k)^2$ then the iterative method (2) has order of convergence four, preserving the order and the optimality of Ostrowski's scheme. As a more general result, if we apply this idea to King's family schemes, which contains Ostrowski's method for a particular value of the parameter (see [5]), we obtain an uniparametric family of optimal derivative-free methods of order four.

**Theorem 1.** *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ in an open interval $I$. If $x_0$ is close enough to $\alpha$, then the iterative method described by*

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f[z_k, x_k]}, \\ x_{k+1} &= y_k - \frac{f(x_k) + \beta f(y_k)}{f(x_k) + (\beta - 2) f(y_k)} \frac{f(y_k)}{f[z_k, x_k]}, \end{aligned} \tag{3}$$