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Numerical solution of threshold problems in epidemics and population dynamics

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ABSTRACT

A new algorithm is proposed for the numerical solution of threshold problems in epidemics and population dynamics. These problems are modeled by the delay-differential equations, where the delay function is unknown and has to be determined from the threshold conditions. The new algorithm is based on embedded pair of continuous Runge–Kutta method of order $p = 4$ and discrete Runge–Kutta method of order $q = 3$ which is used for the estimation of local discretization errors, combined with the bisection method for the resolution of the threshold condition. Error bounds are derived for the algorithm based on continuous one-step methods for the delay-differential equations and arbitrary iteration process for the threshold conditions. Numerical examples are presented which illustrate the effectiveness of this algorithm.

1. Introduction

Denote by $C = C([\alpha, T], \mathbb{R}^m)$ the space of continuous functions from the interval $[\alpha, T]$ into $\mathbb{R}^m$ with the norm defined by

$$|||y|||_{[\alpha,T]} := \sup\left\{e^{-a(t-t_0)}\left\|y\right\|_{[\alpha,t]} : \alpha \le t \le T\right\},$$

where $a > 0$ is a real parameter and $\|y\|_{[\alpha,t]}$ is the uniform norm on the interval $[\alpha, t]$. Let

$$f : [t_0, T] \times C\big([\alpha, T], \mathbb{R}^m\big) \times \mathbb{R}^m \to \mathbb{R}^m$$

and consider the initial-value problem for a functional-differential equation of the form

$$\begin{cases} y'(t) = f\Big(t, y(\cdot), y\Big(t - \tau\big(t, y(\cdot)\big)\Big)\Big), & t \in [t_0, T], \\ y(t) = g(t), & t \in [\alpha, t_0], \end{cases} \tag{1.1}$$

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$\alpha \le t_0$. Problem (1.1) is a generalization of the problem elaborated in [1], where discrete variable methods for its numerical solution are investigated. The function $\tau(t, y(\cdot))$ appearing in this equation is determined from the so-called threshold condition

$$P\Big(t, y(\cdot), \tau\big(t, y(\cdot)\big)\Big) = m, \tag{1.2}$$

with given threshold $m > 0$. Here,

$$P : [t_0, T] \times C\big([\alpha, T], \mathbb{R}^m\big) \times \mathbb{R} \to \mathbb{R}$$

is a given operator. In applications P is usually an integral operator. Observe that (1.2) depends on the unknown function y. The solution to (1.1)–(1.2) will be denoted by $y(t)$ and $\tau(t, y(\cdot))$. Such equations find applications in modeling various problems in epidemics and population dynamics. Specific examples of such problems are presented in Section 2. The existence and uniqueness of the solution to (1.1)–(1.2) are discussed in Section 3.

Since in general, the operator P cannot be computed exactly (P may be an integral operator like in the applications introduced in the next section) we first replace (1.2) by the equation

$$\bar{P}\Big(t, y(\cdot), \tau\big(t, y(\cdot)\big)\Big) = m, \tag{1.3}$$

where $\bar{P}$ is a discrete approximation to P. Next, we describe the numerical approximation to the solution of (1.1), (1.3). Denote by $\bar{y}(t)$ and $\bar{\tau}(t, \bar{y}(\cdot))$ the solution to (1.1), (1.3). To compute numerical approximation $\bar{y}_h$ to $\bar{y}$ we consider the general class of continuous one-step methods of the form

$$\begin{cases} \bar{y}_h(t_n + \theta h_n) = \bar{y}_h(t_n) + h_n \Phi_h\Big(t_n, \theta, \bar{y}_h(\cdot), \bar{y}_h\Big(t_{n+\theta} - \bar{\tau}_h\big(t_{n+\theta}, \bar{y}_h(\cdot)\big)\Big)\Big), & \theta \in (0, 1], \\ \bar{y}_h(t) = g_h(t), & t \in [\alpha, t_0], \end{cases} \tag{1.4}$$

$n = 0, 1, \ldots, N$. Here, $t_{n+1} = t_n + h_n$, $t_{n+\theta} = t_n + \theta h_n$, $n = 0, \ldots, N$, $\theta \in (0, 1]$, with step-sizes h_n which satisfy

$$\sum_{n=0}^{N-1} h_n < T - t_0 \le \sum_{n=0}^{N} h_n. \tag{1.5}$$

Moreover, g_h is an approximation to the initial function g, and $\bar{\tau}_h(t_{n+\theta}, \bar{y}_h(\cdot))$ is an approximation to the solution $\bar{\tau}(t_{n+\theta}, \bar{y}_h(\cdot))$ to the operator equation

$$\bar{P}\Big(t_{n+\theta}, \bar{y}_h(\cdot), \bar{\tau}\big(t_{n+\theta}, \bar{y}_h(\cdot)\big)\Big) = m, \tag{1.6}$$

obtained from (1.3) by replacing t by $t_{n+\theta}$, $y(\cdot)$ by $\bar{y}_h(\cdot)$ and $\tau(t, y(\cdot))$ by $\bar{\tau}(t_{n+\theta}, \bar{y}_h(\cdot))$. In this formulation the increment function Φ_h and the operator equation (1.6) depend on $\bar{y}_h(\cdot)$ and $\bar{y}_h(t_{n+\theta} - \bar{\tau}_h(t_{n+\theta}, \bar{y}_h(\cdot)))$ although in practical applications this dependence is usually restricted to a discrete set of values such as, for example, $\bar{y}_h(t_{n+c_i})$ and $\bar{y}_h(t_{n+c_i} - \bar{\tau}_h(t_{n+c_i}, \bar{y}_h(\cdot)))$, $i = 1, 2, \ldots, s$, where c_i are given abscissas usually chosen from the interval $[0, 1]$. This is the case for continuous Runge–Kutta methods considered in Section 5.

Depending on the form of the increment function Φ_h the formulation (1.4) includes both the explicit and implicit formulas for (1.1), (1.3). Note that, although $\bar{y}_h(t_{n+\theta})$ is computed from (1.4) and the quantity $\bar{\tau}\big(t_{n+\theta}, \bar{y}_h(\cdot)\big)$ is computed from (1.6), these equations are not independent and (1.6) has to be resolved at each time step of numerical integration for the method (1.4).

We are interested to estimate the global error $y - \bar{y}_h$, where y is the solution to (1.1) with $\tau(t, y(\cdot))$ given by (1.2), and $\bar{y}_h$ is computed from (1.4) with the approximation $\bar{\tau}_h(t_n, \bar{y}_h(\cdot))$ to $\bar{\tau}(t_n, \bar{y}_h(\cdot))$ computed by some iterative procedure applied to Eq. (1.6). This error consists of two parts:

$$y - \bar{y}_h = \big(y - \bar{y}\big) + \big(\bar{y} - \bar{y}_h\big),$$

and we have

$$\big\|y - \bar{y}_h\big\|_{[\alpha,T]} \le \big\|y - \bar{y}\big\|_{[\alpha,T]} + \big\|\bar{y} - \bar{y}_h\big\|_{[\alpha,T]}. \tag{1.7}$$

Here, for $x \in C([\alpha, t], \mathbb{R}^m)$ and $t \in [\alpha, T]$ the norm $\|x\|_{[\alpha,t]}$ is defined by

$$\big\|x\big\|_{[\alpha,t]} := \sup\Big\{\big\|x(s)\big\| : \alpha \le s \le t\Big\},$$

where $\|\cdot\|$ is any norm on $\mathbb{R}^m$. The first term on the right hand side of the above inequality will be investigated in Section 4 using the theory of integral inequalities. The second term on the right hand side of (1.7) will be investigated in Section 5 using the generalization of the theory of one-step methods for functional differential equations. In Section 6 we describe the adaptation of continuous Runge–Kutta methods for ordinary differential equations to the problem (1.1), (1.3). In Section 7 we describe the numerical algorithm for the solution of (1.1), (1.3) based on embedded pair of continuous Runge–Kutta methods of order $p = 4$ and discrete method of order $q = p - 1 = 3$ which is used for error estimation. In this section