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Building blocks for designing arbitrarily smooth subdivision schemes with conic precision

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HIGHLIGHTS

- Non-stationary extension of Lane–Riesenfeld algorithm.
- New family of alternating primal/dual subdivision schemes reproducing conics.
- New family of non-stationary interpolatory $2n$-point schemes reproducing conics.
- Explicit formulation and recurrence relations.
- Analysis of the main properties of the above families.

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ABSTRACT

Since subdivision schemes featured by high smoothness and conic precision are strongly required in many application contexts, in this work we define the building blocks to obtain new families of non-stationary subdivision schemes enjoying such properties. To this purpose, we firstly derive a non-stationary extension of the Lane–Riesenfeld algorithm, and we exploit the resulting class of schemes to design a non-stationary family of alternating primal/dual subdivision schemes, all featured by reproduction of $\{1, x, e^{tx}, e^{-tx}\}$, $t \in [0, \pi) \cup i\mathbb{R}^+$. Then, we focus our attention on interpolatory subdivision schemes with conic precision, that can be obtained as a byproduct of the above classes. In particular, we present a novel construction of a family of non-stationary interpolatory $2n$-point schemes which generalizes the well-known Dubuc–Deslauriers family in such a way the nth ($n \geq 2$) family member reproduces $\Pi_{2n-3} \cup \{e^{tx}, e^{-tx}\}$, $t \in [0, \pi) \cup i\mathbb{R}^+$, and keeps the original smoothness of its stationary counterpart unchanged.

1. Introduction

Subdivision schemes are efficient tools for generating smooth curves and surfaces as the limit of an iterative algorithm based on simple refinement rules. More precisely, in the univariate case, for any given set of initial control points $\mathbf{P}^{(0)} := \{P_i^{(0)},\ i \in \mathbb{Z}\}$, a linear subdivision scheme recursively produces denser sets of control points $\mathbf{P}^{(k+1)}$, for all $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, by computing local linear combinations of points from the previous level. If the same refinement rules are used at all levels of refinement, then the scheme is called *stationary*, otherwise *non-stationary*.

In the stationary context, the Lane–Riesenfeld algorithm [1] defines the symbols associated to the family of B-spline schemes of order ℓ, with $\ell \in \mathbb{N}$. In literature, the use of these symbols as 'building blocks' to define both interpolatory

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schemes [2] and subdivision schemes with enhanced reproduction capabilities [3] has been recently shown. In fact, Conti and Romani observed that ℓ-point (with ℓ even) Dubuc–Deslauriers schemes [4] are characterized by a symbol containing the factor $\frac{(z+1)^\ell}{2^{\ell-1}}$, while Hormann and Sabin noticed that the same factor (with $\ell \in \mathbb{N}$) is also contained in the symbol of the family of subdivision schemes with cubic precision. The latter family is indeed defined by the product of the symbol of the Lane–Riesenfeld's family with a degree-2 polynomial, that they called *kernel*, tailored to increase the degree of polynomial reproduction of B-spline schemes from one to three. Moreover, in [5] it has been also recently illustrated that the first member of the Lane–Riesenfeld's family and that of the Hormann–Sabin's family can be combined together to give rise to a recursive formula defining the interpolatory $2n$-point Dubuc–Deslauriers schemes for all $n \geq 3$.

These observations prompted us to study the generalization of these two fundamental classes of schemes to the non-stationary setting. Our first contribution in this direction consists in the proposal of a level-dependent extension of the Lane–Riesenfeld algorithm, aimed at providing the symbols of normalized exponential B-splines. These symbols, together with a non-stationary version of Hormann–Sabin's kernels, are successively used as 'building blocks' to define a family of alternating primal/dual subdivision schemes reproducing conics. The first member of the resulting family, combined with the first one of the novel Lane–Riesenfeld's family, is shown to originate a three-term recurrence formula defining the symbols of the non-stationary interpolatory $2n$-point schemes reproducing the space span$\{1, x, \dots, x^{2n-3}, e^{tx}, e^{-tx}\}$, where $t \in [0, \pi) \cup \mathrm{i}\mathbb{R}^+$ and $n \in \mathbb{N},\ n \geq 3$. We highlight the fact that, non-stationary subdivision schemes enjoying properties like interpolation, conic precision and arbitrarily high smoothness, are considered wished tools both in geometric modelling and image segmentation. As to the latter, we recall that one of the most used tools for efficient image segmentation are active contours (snakes), i.e. 2D curves evolving through the image, capable of perfectly outlining elliptic objects and offering user-friendly models, versatile enough to provide a close smooth approximation of any closed polyline in the plane [6].

The remainder of the paper is organized as follows. In Section 2 we start by presenting all the fundamental notions about stationary and non-stationary subdivision schemes that are necessary to the development of the subsequent results. Section 3 is devoted to the stationary context. After recalling the basic formulations of the Lane–Riesenfeld algorithm and the Hormann–Sabin's family, we review the existing different formulations of the family of $2n$-point interpolatory Dubuc–Deslauriers schemes, and we show how to obtain its symbol exploiting the Lane–Riesenfeld's and Hormann–Sabin's families as building blocks. All remaining sections deal with the non-stationary setting and present original results. In particular, after presenting our extension of the Lane–Riesenfeld algorithm (Section 4), we construct a family of alternating primal/dual non-stationary subdivision schemes reproducing conics, which generalizes the well-known Hormann–Sabin's family (Section 5). Finally, in Section 6 we exploit a suitable perturbation of the symbols of the well-known Dubuc–Deslauriers schemes to define non-stationary interpolatory $2n$-point schemes which achieve the property of conic precision, without affecting the smoothness order of the original proposal.

2. Background notions

2.1. The stationary case

Let $\mathrm{a}_i,\ i \in \mathbb{Z}$, be the coefficients appearing in the linear combination that defines at each iteration the new-level points. Then, for each $k \in \mathbb{N}_0$, the refinement rules are

$$P_{2i+h}^{(k+1)} = \sum_{j \in \mathbb{Z}} \mathrm{a}_{2j+h} P_{i-j}^{(k)}, \quad h = 0, 1. \tag{2.1}$$

The set of coefficients $\{\mathrm{a}_i \in \mathbb{R},\ i \in \mathbb{Z}\}$ appearing in (2.1) is called *subdivision mask* and is denoted by $\mathbf{a}$. The subdivision scheme with mask $\mathbf{a}$ is denoted by $\mathcal{S}_{\mathbf{a}}$ and can be equivalently seen as the repeated application of the subdivision matrix $M = \{M(i,j) = \mathrm{a}_{i-2j} : i, j \in \mathbb{Z}\}$ to the initial data $\mathbf{P}^{(0)}$.

Applying the z-transform, we can associate the mask $\mathbf{a}$ to the Laurent series

$$A(z) = \sum_{i \in \mathbb{Z}} \mathrm{a}_i z^i, \quad z \in \mathbb{C} \setminus \{0\}, \tag{2.2}$$

which is called the *symbol* of the subdivision scheme. Since only a finite number of coefficients a_i are non-zero, the Laurent series $A(z)$ is indeed a Laurent polynomial.

The symbol $A(z)$ has been shown to be a convenient tool to investigate both convergence/smoothness and generation/reproduction properties of the subdivision scheme $\mathcal{S}_{\mathbf{a}}$.

We recall that a subdivision scheme is said to be *convergent* if, for any initial data $\mathbf{P}^{(0)} \in \ell^\infty(\mathbb{Z})$, there exists a function $\mathcal{F} \in C^0(\mathbb{R})$ such that for any compact set Ω in $\mathbb{R}$, $\lim_{k\to+\infty} \sup_{i \in \mathbb{Z} \cap 2^k\Omega} |P_i^{(k)} - \mathcal{F}(2^{-k}i)| = 0$, and $\mathcal{F}$ is not identically 0 for some initial data $\mathbf{P}^{(0)}$. In particular, for any convergent subdivision scheme, we denote by Φ the limit function obtained from the initial sequence $\boldsymbol{\delta} = \{\delta_{i,0} : i \in \mathbb{Z}\}$ where $\delta_{i,0} = \begin{cases} 1, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$ Φ is usually called the *basic limit function* of the subdivision scheme.

Existing results on polynomial generation and reproduction properties of a stationary subdivision scheme are restricted to the class of non-singular schemes, i.e. the ones that generate the zero function if and only if $\mathbf{P}^{(0)}$ is the zero sequence. In