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Comparison of boundedness and monotonicity properties of one-leg and linear multistep methods

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ABSTRACT

One-leg multistep methods have some advantage over linear multistep methods with respect to storage of the past results. In this paper boundedness and monotonicity properties with arbitrary (semi-)norms or convex functionals are analyzed for such multistep methods. The maximal stepsize coefficient for boundedness and monotonicity of a one-leg method is the same as for the associated linear multistep method when arbitrary starting values are considered. It will be shown, however, that combinations of one-leg methods and Runge–Kutta starting procedures may give very different stepsize coefficients for monotonicity than the linear multistep methods with the same starting procedures. Detailed results are presented for explicit two-step methods.

1. Introduction

The ODE systems and basic assumption

We consider systems of ordinary differential equations (ODEs) with given initial value in a vector space $\mathbb{V}$,

$$u'(t) = F(u(t)), \qquad u(0) = u_0, \tag{1.1}$$

where $F : \mathbb{V} \to \mathbb{V}$ and $u_0 \in \mathbb{V}$. Throughout this paper we will make the following basic assumption: there is a constant $\tau_0 > 0$ such that

$$\|v + \tau_0 F(v)\| \leq \|v\| \quad \text{for all } v \in \mathbb{V}, \tag{1.2}$$

where $\|\cdot\|$ denotes a norm, seminorm, or convex functional on $\mathbb{V}$ (cf. Section 2).

It is easy to see that (1.2) implies $\|v + \Delta t\, F(v)\| \leq \|v\|$ for all $\Delta t \in (0, \tau_0]$. Consequently, applying the forward Euler method $u_n = u_{n-1} + \Delta t\, F(u_{n-1})$, $n \geq 1$, with stepsize $\Delta t > 0$ to compute approximations $u_n \approx u(t_n)$ at $t_n = n\Delta t$, we obtain $\|u_n\| \leq \|u_0\|$ for $n \geq 1$ under the stepsize restriction $\Delta t \leq \tau_0$. For general one-step methods, this property under a stepsize restriction $\Delta t \leq \gamma\, \tau_0$ is often referred to as *monotonicity* or *strong stability preservation* (SSP). For multistep methods this can be generalized in several ways, which will be addressed in this paper.

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Linear multistep and one-leg methods

In this paper we will consider one-leg and linear multistep methods for finding the approximations $u_n \approx u(t_n)$ at the step points $t_n = n\Delta t$, $n \geq 1$. It is supposed that starting vectors $u_0, u_1, \ldots, u_{k-1} \in \mathbb{V}$ are known.

A linear multistep (LM) method applied to (1.1) reads

$$u_n = \sum_{j=1}^{k} a_j u_{n-j} + \Delta t \sum_{j=0}^{k} b_j F(u_{n-j}) \tag{1.3}$$

for $n \geq k$. The parameters a_j, b_j and $k \in \mathbb{N}$ define the method. Along with this linear multistep method, we also consider the corresponding k-step one-leg (OL) method

$$u_n = \sum_{j=1}^{k} a_j u_{n-j} + \Delta t \beta F(v_n), \qquad v_n = \sum_{j=0}^{k} \hat{b}_j u_{n-j} \tag{1.4}$$

for $n \geq k$, where $\hat{b}_j = b_j/\beta$ and $\beta = \sum_{j=0}^{k} b_j \neq 0$. If $b_0 = 0$ these multistep methods are called explicit, and if $b_0 \neq 0$ they are called implicit.

One-leg methods were introduced by Dahlquist [1], originally only to facilitate the analysis of linear multistep methods. Subsequently, it was realized that one-leg methods might be useful on their own, not just as an analysis tool. It is known that the conditions for consistency of order p are the same if $p = 1, 2$, but for larger p the one-leg method has to satisfy more order conditions than the corresponding linear multistep method; cf. [2], for instance.

On the other hand, one-leg methods have an advantage over the corresponding linear multistep methods with respect to storage, which is often important for large-scale problems when function evaluations of F are expensive. If, for example, $b_0 = 0$ but $a_k, b_k \neq 0$, then for a step (1.3) with the linear multistep method we need storage of the vectors $u_{n-1}, \ldots, u_{n-k}$ and $F(u_{n-2}), \ldots, F(u_{n-k})$, together with an evaluation of $F(u_{n-1})$. For a step (1.4) with the one-leg method only storage of $u_{n-1}, \ldots, u_{n-k}$ is needed, together with evaluation of $F(v_n)$.

Scope of the paper

In this paper we will first consider the property

$$\|u_n\| \leq \mu \cdot \max_{0 \leq j < k} \|u_j\| \quad \text{for all } n \geq k \text{ and } 0 < \Delta t \leq \gamma\, \tau_0, \tag{1.5}$$

whenever the basic assumption (1.2) is satisfied. Here the factor $\mu \geq 1$ and the stepsize coefficient $\gamma \geq 0$ are determined by the multistep method, and we are interested in having $\gamma > 0$ as large as possible. If (1.5) holds with $\mu = 1$, then this property will be called *monotonicity*. For many interesting methods, this property (1.5) will only hold with some $\mu > 1$, in which case we refer to it as *boundedness*.

It is known, see e.g. [3,4], that the condition for monotonicity for either the linear multistep method (1.3) or the one-leg method (1.4) reads

$$a_j \geq \gamma\, b_j \geq 0 \quad (1 \leq j \leq k). \tag{1.6}$$

This requires that all coefficients of the method are non-negative, which severely restricts the class of methods. It is therefore of interest to study more relaxed properties.

The boundedness property (1.5) with some $\mu \geq 1$, has been studied for linear multistep methods. Sufficient stepsize conditions $\Delta t \leq \gamma \tau_0$ were derived in [5,6] for having (1.5) with arbitrary seminorms under the basic assumption (1.2). More simple conditions were found in [7], and these conditions were shown be necessary as well as sufficient.

In (1.5) the starting values $u_1, \ldots, u_{k-1}$ are arbitrary. In practice these starting values will be computed from the given initial value u_0, for instance by a Runge–Kutta method. For such combinations of multistep methods and Runge–Kutta starting procedures the following monotonicity property

$$\|u_n\| \leq \|u_0\| \quad \text{for all } n \geq 1 \text{ and } 0 < \Delta t \leq \gamma\, \tau_0, \tag{1.7}$$

can still be valid, even if the multistep method itself is not monotone, but only bounded for arbitrary starting values, that is, (1.5) is valid with $\mu > 1$, not with $\mu = 1$.

For some combinations of linear multistep methods and Runge–Kutta starting procedures, the monotonicity property (1.7) was studied in [7], where conditions were derived with arbitrary seminorms and nonnegative sublinear functionals. Earlier, for some two-step methods, sufficient conditions with seminorms were found in [5].

In this paper we will first describe in Section 2 a general framework for having boundedness with arbitrary starting vectors, or monotonicity with starting procedures. This framework, which is valid for general multistep multistage methods, will be based on the approach of Spijker [4] for monotonicity, and of Hundsdorfer, Mozartova and Spijker [8] for boundedness. The results will then be applied to linear multistep methods and one-leg methods. For this, the methods will be formulated in Section 3 in terms of input and output processes, so that the general framework is applicable.