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Variational localizations of the dual weighted residual estimator

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ABSTRACT

The dual weighted residual method (DWR) and its localization for mesh adaptivity applied to elliptic partial differential equations are investigated. The contribution of this paper is twofold: first, we introduce a novel localization technique based on the introduction of a partition of unity. This new technique is very easy to apply, as neither strong residuals nor jumps over element edges are required. Second, we compare and analyze (theoretically and numerically) different localization techniques used for mesh adaptivity with respect to their effectivity. Here, we focus on localizations in variational formulations that do not require the evaluation of the corresponding differential operator in the classical strong formulation. In our mathematical analysis, we show for different localization techniques (established methods and our new approach) that the local error indicators used for mesh adaptivity converge with proper order in the error functional. Several numerical tests substantiate our theoretical investigations.

1. Introduction

In this work, we investigate the dual weighted residual method (DWR) and its localization for mesh adaptivity applied to elliptic partial differential equations. Our goal is twofold: first, and most important, we introduce a new localization technique, given in weak form that avoids both the evaluation of strong residuals and jump terms over element edges. This method is easy to implement and therefore suitable for coupled multiphysics systems with many different equations. The second aim is then to analyze different localization techniques with respect to their effectivity. For some established localization techniques, this has not yet been accomplished.

The DWR method allows for estimating the error $u - u_h$ between the exact solution $u \in V$ (for a function space V) of the PDE and its Galerkin solution $u_h \in V_h \subset V$ in general (error) functionals $J : V \to \mathbb{R}$. These functionals can be norms but also more general expressions, like point-values, (local) averages or technical expressions like (in the case of fluid dynamics) lift- or drag-coefficients. Error estimators based on the DWR method always consist of residual evaluations, that are weighted by adjoint sensitivity measures. These sensitivities are the solution to adjoint problems that measure the influence of the error functional J.

The DWR method goes back to Becker and Rannacher [1,2] and is based on the pioneering work by Eriksson, Estep, Hansbo and Johnson [3]. It has been further developed by various researchers [4–6] and has been applied to a vast number of application problems including fluid-dynamics [7], structural dynamics, and further to complex multiphysics problems

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like chemically reactive flows [8] or fluid–structure interactions [9–13]. A completely different field, where the use of strong residuals is to be avoided is kinetic theory, see [14] for an application of goal oriented error estimation to Boltzmann-type equations.

In this contribution to the DWR method, we focus on more principle questions that arise in its application. First, the adjoint weights entering the error estimator usually must be approximated, as they involve the unknown exact solution $z \in V$ of an adjoint problem. Section 3 provides an overview of different approximation techniques commonly used. Second, if used for adaptive mesh refinement, the error estimator $\eta_h \approx J(u) - J(u_h)$ must be localized to positive error indicators

$$|\eta_h| \approx \sum_i |\eta_i|, \tag{1}$$

which describe the local error contribution $|\eta_i|$ of a mesh element or a mesh node, and that can be used to establish adaptive mesh refinement schemes. In the central Section 4, we describe different localization techniques for the DWR estimator and discuss their effectivity: a localization is called effective, if the sum of local indicators do not overestimate the error. Error indicators, that highly overestimate the error will lead to adaptive meshes, which do not fit to the problem. Usually, for adaptive methods, one aims at showing effectivity, such that the estimator bounds the error from below and above

$$c_1 \sum_i |\eta_i| \le \|\nabla(u - u_h)\| \le c_2 \sum_i |\eta_i|. \tag{2}$$

We cannot expect such a sharp result, as we are not looking at norms only, but at errors in general functionals $J(\cdot)$. The DWR estimator is an error approximation $\eta_h \approx J(u) - J(u_h)$, but usually not a rigorous estimate. Usually, it is straightforward to bound the indicators by the estimator from below

$$|\eta_h| \le \sum_i |\eta_i|. \tag{3}$$

The main contribution of this work is to provide insight to the opposite direction. We are not able to bound the sum of indicators by the estimator $|\eta_h|$ or even the functional error $|J(u)-J(u_h)|$ itself, but we can show, that the error $|J(u)-J(u_h)|$ and the indicators $\sum_i |\eta_i|$ satisfy a common upper bound. This has not been accomplished for some commonly used localization techniques. Finally, we introduce a novel localization technique, that is strikingly simple in its application and also permits a very simple proof to show the effectivity of indicators (within the limits just discussed).

In Section 5, several numerical test cases are presented to discuss the performance of the different localization strategies. Finally, in Section 6, we conclude with some further remarks.

Let us begin in the following Section 2 by gathering the notation and shortly introducing the dual weighted residual method for error estimation.

2. The dual weighted residual method for error estimation

By $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$ we denote a domain with polygonal or polyhedral domain. On Ω, we denote by $(\cdot,\cdot)$ the L^2-inner product and by $\|\cdot\|$ the corresponding L^2-norm. By $H^{r+1}(\Omega)$ we denote the space of Lebesgue functions with square integrable weak derivatives up to degree $r+1$. In particular, by $V := H_0^1(\Omega)$ we denote the space of $H^1(\Omega)$ functions with trace zero on the boundary $\partial\Omega$.

2.1. DWR for the Poisson problem and linear goal functionals

By $u \in V$ we denote the solution of the Poisson equation

$$(\nabla u, \nabla \phi) = (f, \phi) \quad \forall \phi \in V, \tag{4}$$

for a given right hand side function $f \in H^{-1}(\Omega)$. We consider the case of homogeneous Dirichlet boundary conditions on $\partial\Omega$ only. Next, we denote by $V_h := V_h^{(r)} \subset V$ a finite dimensional, piece-wise polynomial of degree r finite element subspace and by $u_h \in V_h$ the finite element solution

$$(\nabla u_h, \nabla \phi_h) = (f, \phi_h) \quad \forall \phi_h \in V_h. \tag{5}$$

Here we only consider finite element spaces of polynomial degree $r \ge 1$ on shape-regular triangulations Ω_h, such that there exists an interpolation operator $i_h : V \to V_h$ on every element $K \in \Omega_h$

$$\|\nabla^k(u - i_h u)\|_K \le c_{\text{in}} h_K^{r+1-k} \|\nabla^{r+1} u\|_K \quad \forall u \in H^{r+1}(K),\ k = 0, 1, 2, \tag{6}$$

with $h_K := \text{diam}(K)$ and where $\|\cdot\|_K$ is the L^2-norm on K. The interpolation constant c_{in} depends on the polynomial degree r and the triangulation Ω_h. Further, on element boundaries ∂K, we use the estimate

$$\|u - i_h u\|_{\partial K} \le c_{\text{in}} h^{r+\frac{1}{2}} \|\nabla^{r+1} u\|_K, \tag{7}$$

with $h = \max_K h_K$. Adaptive meshes are realized with hanging nodes, see [15] for details on the construction.