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Optimal error bound and simplified Tikhonov regularization method for a backward problem for the time-fractional diffusion equation

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ABSTRACT

In this paper, we consider a backward problem for a time-fractional diffusion equation. Such a problem is ill-posed. The optimal error bound for the problem under a source condition is analyzed. A simplified Tikhonov regularization method is utilized to solve the problem, and its convergence rates are analyzed under an a priori regularization parameter choice rule and an a posteriori regularization parameter choice rule, respectively. Numerical examples show that the proposed regularization method is effective and stable, and both parameter choice rules work well.

1. Introduction

Fractional calculus is almost as old as its more familiar integer-order counterparts [1,2]. It has been widely used to describe a range of problems in research areas as diverse as physics [3], finance [4–7], hydrology [8,9] and so on. The most important advantage of using fractional differential equations in these and other applications is its non-local property which is different from the integer order differential operator which is so called local operator. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. This is why fractional calculus has become more and more popular, as it is more realistic.

The time-fractional PDEs describe the continuous time random walk phenomenon. The backward problem of diffusion process is of great importance in engineering and aims at detecting the previous status of physical field from its present information. In general, no solution which satisfies the diffusion equation with final data and the boundary conditions exists. Even if a solution exists, it does not depend continuously on the final data and any small perturbation in the given data may cause large change to the solution. So we need some regularization methods to deal with this problem.

In this paper, we consider the backward problem for a time-fractional diffusion equation

$$\begin{cases} D_t^{\alpha} u(x,t) = u_{xx}(x,t), & (x,t) \in (0,\pi) \times (0,T), \\ u(0,t) = u(\pi,t) = 0, & t \in [0,T], \\ u(x,T) = g(x), & x \in [0,\pi], \end{cases} \tag{1.1}$$

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where D_t^α is the Caputo fractional derivative of order α ($0 < \alpha \le 1$) defined by

$$D_t^\alpha u(x,t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_s(x,s)}{(t-s)^\alpha} ds, & 0 < \alpha < 1, \\ u_t(x,t), & \alpha = 1, \end{cases} \tag{1.2}$$

where $\Gamma(\cdot)$ is Gamma function and $g(x)$ is the final status satisfying the compatibility condition: $g(0) = g(\pi) = 0$.

Our backward problem is to find an approximation to the temperature $u(x,t)$ for $t \in [0,T)$ from the measurement $g^\delta(x)$ which is a noise-contaminated function for the exact temperature $g(x)$ and satisfies:

$$\|g^\delta(x) - g(x)\| \le \delta, \tag{1.3}$$

where $\|\cdot\|$ refers to the L^2 norm and $\delta > 0$ is a noise level.

For $\alpha = 1$, it is a classical ill-posed problem and has been studied by many researchers. Various approaches have been investigated and applied to this problem [10–20]. Zheng and Wei solved a space-fractional backward diffusion problem in [21]. But to our knowledge, there are very few works on the backward problem for the time-fractional diffusion equation. Liu [22] used a quasi-reversibility method to solve the backward problem (1.1). Sakamoto and Yamamoto [23] proved that there exists a unique weak solution for the backward problem (1.1) under the condition $g(x) \in H^2(0,\pi) \cap H_0^1(0,\pi)$. In this paper, we focus on the case $0 < \alpha < 1$.

By the separation of variables [24], we can easily obtain the solution to (1.1) as

$$u(x,t) = \sum_{n=1}^{\infty} \frac{E_{\alpha,1}(-\lambda_n t^\alpha)}{E_{\alpha,1}(-\lambda_n T^\alpha)} g_n X_n, \tag{1.4}$$

where $\lambda_n = n^2$ is the eigenvalues of the operator $L = -\frac{\partial^2}{\partial x^2}$, $X_n = \sqrt{\frac{2}{\pi}} \sin nx$ is corresponding eigenfunctions, $g_n = \int_0^\pi g(x) X_n dx$, $n = 1, 2, \ldots$, and $E_{\alpha,\beta}(z)$ is Mittag-Leffler function [2] which is defined as follows:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

and possesses the following properties.

Proposition 1.1 ([22]). *Assume that $0 < \alpha_0 < \alpha_1 < 1$. Then there exist constants $C_\pm > 0$, depending only on α_0, α_1 such that for all $\alpha \in [\alpha_0, \alpha_1]$, we have*

$$\frac{C_-}{\Gamma(1-\alpha)} \frac{1}{1-x} \le E_{\alpha,1}(x) \le \frac{C_+}{\Gamma(1-\alpha)} \frac{1}{1-x}, \quad \textit{for all } x \le 0. \tag{1.5}$$

Unless otherwise specified, we assume $\alpha \in [\alpha_0, \alpha_1]$ for some α_0, α_1 in this paper.

If the data is noise-contaminated, then

$$u^\delta(x,t) = \sum_{n=1}^{\infty} \frac{E_{\alpha,1}(-\lambda_n t^\alpha)}{E_{\alpha,1}(-\lambda_n T^\alpha)} g_n^\delta X_n, \tag{1.6}$$

where

$$g_n^\delta = \int_0^\pi g^\delta(x) X_n dx. \tag{1.7}$$

From Proposition 1.1, it is easy to check that this backward problem is well-posed for $t \in (0,T)$. But at $t = 0$, because $\frac{1}{E_{\alpha,1}(-\lambda_n T^\alpha)} \ge \frac{\Gamma(1-\alpha)}{C_+}(1 + \lambda_n T^\alpha) = \frac{\Gamma(1-\alpha)}{C_+}(1 + n^2 T^\alpha)$, when we calculate $u(x,0)$ from $g^\delta(x)$, the small error in the high-frequency components of $g^\delta(x)$ will be amplified by the factor $\frac{1}{E_{\alpha,1}(-\lambda_n T^\alpha)}$, thus this backward problem is ill-posed. We must use some regularization methods to deal with this problem.

We will use a simplified Tikhonov regularization method to regularize our backward problem in this study. This method was first introduced by Carasso in [25], then the idea of this method has been successfully used for solving various types of ill-posed problems [26–28]. In order to examine whether a regularization method is optimal, we will give the optimal error bound for the problem under a source condition.

The paper is organized as follows. In Section 2, we give some preliminary results. Then, we apply these results to problem (1.1) and give the optimal error bound in Section 3. In Section 4, we propose a simplified Tikhonov regularization method and give a convergence estimate under an a priori assumption for the exact solution and an a priori regularization parameter choice rule. In Section 5, we give a convergence estimate under an a posteriori regularization parameter choice rule. Finally the numerical examples are given in Section 6 for showing that our method is effective and stable. A short conclusion in Section 7 summarizes the content of this paper.