Journal of Computational and Applied Mathematics 279 (2015) 293–305

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A shooting reproducing kernel Hilbert space method for multiple solutions of nonlinear boundary value problems

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ARTICLE INFO

Article history:
Received 19 March 2014
Received in revised form 8 October 2014

Keywords:
Multiple solutions
Reproducing kernel Hilbert space
Shooting method
Nonlinear Bratu equation
Nonlinear reaction–diffusion equation
Nonlinear three-point boundary value problem

ABSTRACT

In this work an iterative method is proposed to predict and demonstrate the existence and multiplicity of solutions for nonlinear boundary value problems. In addition, the proposed method is capable of calculating analytical approximations for all branches of solutions. This method is a combination of reproducing kernel Hilbert space method and a shooting-like technique which takes advantage of two powerful methods for solving nonlinear boundary value problems. The formulation and implementation of this iterative method is discussed for nonlinear second order with two and three-point boundary value problems. Also, the convergence of the proposed method is proved. To demonstrate the computational efficiency, the mentioned method is implemented for some nonlinear exactly solvable differential equations including strongly nonlinear Bratu equation and nonlinear reaction–diffusion equation. It is also applied successfully to two nonlinear three-point boundary value problems with unknown exact solutions. In the last example a new branch of solutions is found which shows the power of the method to search for multiple solutions and indicates that the method may be successful in cases where purely analytic methods are not.

1. Introduction

Boundary value problems for ordinary differential equations play an essential role in both theory and applications. They are used to describe a large amount of physical, biological and chemical phenomena. Finding approximate analytic solutions of nonlinear boundary value problems is extremely significant in engineering and physical sciences. Here, our concern is the existence of solutions, prediction of the number of solutions, properties of solutions and calculation of the approximate analytic solutions of nonlinear second order two and three point boundary value problems. Nonlinear boundary value problems (BVPs) may have no, one or more than one solutions [1–4]. There are many methods to give approximate solutions of nonlinear BVPs, but despite the existence of multiple solutions for some of these problems, mentioned methods usually converge to only one branch of the solutions. It is important to predict the multiplicity of solutions and also approximate all branches of the solutions sufficiently. Abbasbandy et al. [1,2] proposed a method based on homotopy analysis method to predict the multiplicity of solutions of nonlinear problems. Liao applied the homotopy analysis method to obtain new branches of solutions of nonlinear problems [3,4]. The aim of this paper is to introduce the technique based on reproducing kernel Hilbert space methods combined with a shooting like technique, to predict the multiplicity of solutions and to

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http://dx.doi.org/10.1016/j.cam.2014.11.014
0377-0427/

calculate all branches of multiple solutions for nonlinear BVPs. Reproducing kernel theory has important applications in numerical analysis. Recently, a lot of research works have been devoted to the developments and applications of reproducing kernel space methods, such as numerical solution of PDEs via collocation and Galerkin approaches [5,6], approximation of stochastic partial differential equations [7], numerical solution of integral equations [8,9] and applications to Machine Learning [10]. The reproducing kernel Hilbert space methods have successfully been applied to several nonlinear problems, such as nonlinear system of boundary value problems, nonlinear initial value problems, singular nonlinear two-point periodic boundary value problems and singularly perturbed turning point problems [11–15]. But the convergence theorem of the mentioned methods demanded the existence and uniqueness of solutions of nonlinear boundary value problems and cannot be applied directly for the problems with multiplicity in solutions. We will overcome this difficulty by solving certain related initial value problems, due to the fact that initial value problems are much better understood than boundary value problems. The best known and most applicable initial value method is the shooting method. Combination of the two methods gives us a powerful iterative technique for nonlinear second order two and three point boundary value problems which can be easily adopted for higher order problems. The convergence of the proposed iterative method will be proven too. To be more specific, we consider here the second order nonlinear BVPs of the form

$$u'' = f(x, u, u'); \quad x \in [a, b], \tag{1.1}$$

under one of the following types of boundary conditions, with two-point

$$u(a) = u_0, \qquad u(b) = u_1, \tag{1.2}$$

$$u'(a) = u_0, \qquad u(b) = u_1, \tag{1.3}$$

or with three-point

$$u(a) = u_0, \qquad u(b) = \rho u(\eta), \tag{1.4}$$

where both ρ and η are given constants. Numerical examples are given to demonstrate the computation efficiency and power of the method. The method is implemented for some nonlinear exactly solvable differential equations including strongly nonlinear Bratu equation [16–18] and nonlinear reaction–diffusion equation [1,19,20]. To demonstrate the efficiency and power of our method, we applied it to two nonlinear three-point boundary value problems with unknown exact solutions and unknown existence, uniqueness or multiplicity of solutions. The obtained results are in full agreement with the results reported in [21]. In addition, more information about existence and multiplicity are obtained and further, we approximate accurately all branches of solutions. In the last example we obtain a new branch of solutions which shows the advantages of the proposed method in this manuscript versus the analytical methods such as [21]. Simplicity and applicability are important advantages of our method in practice, which avoids complicated assumptions and conditions on the problems, like the purely analytical methods, such as lower and upper solutions [22–24].

2. Description of the method

For the nonlinear boundary value problems such as (1.1) with either boundary conditions (1.2), (1.3) or (1.4) the procedure stated as follows. We replace the boundary value Problem by initial value problem,

$$\begin{cases} u'' = f(x, u, u'); \quad x \in [a, b], \\ u(a) = u_0, \qquad u'(a) = \alpha, \end{cases} \tag{2.5}$$

where α is an unknown to be determined in such a way that the other boundary condition $u(b) = u_1$ (for simplicity we describe the method with boundary condition (1.2) and the other cases are treated in a similar way). We assume here that the solutions of the initial value problems (2.5) exist, is unique, and depend continuously on their initial conditions. The existence and uniqueness of the solution of initial value problems are widely investigated in the literature for example see [25–27]. We must now approximate the solution of initial value problem (2.5) with sufficient accuracy and also α has to be chosen such that the solution $u(x; \alpha)$ satisfies the boundary condition. Because of the nonlinearity the solution of the initial value problem is not readily obtainable. We will use the *RKHS* method for calculating the accurate approximation of the initial value problem (2.5) and for avoid the difficulty of the parametric calculation in iterative *RKHS* method, we will use instead of one value of α, a sequence $\alpha_n \to \alpha^*$ such that $u(b; \alpha_n) \to u_1$ as $n \to \infty$. For a determined α the approximated solution of (2.5) can be obtained as follows. Put $Lu \equiv u''(x)$, after homogenization (such a homogenization can be found in [11]), the problem (2.5) can be convert into the following form:

$$\begin{cases} Lu = f(x, u, u'; \alpha); \quad x \in [a, b], \\ u(a) = 0, \qquad u'(a) = 0. \end{cases} \tag{2.6}$$

In order to solve problem (2.6), reproducing kernel spaces $W_2^3[a, b]$ and $W_2^1[a, b]$ are defined in the following, for more details and proofs we refer to [11]. For simplicity here we let $a = 0$ and $b = 1$.