Journal of Computational and Applied Mathematics 272 (2014) 41–56

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Variational time discretization methods for optimal control problems governed by diffusion–convection–reaction equations

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ARTICLE INFO

Article history:
Received 3 February 2013
Received in revised form 2 May 2014

MSC:
49N10
49K20
65M60
65M15

Keywords:
Optimal control problems
Unsteady diffusion–convection–reaction equation
Variational time discretization
A priori error estimates

ABSTRACT

In this paper, the distributed optimal control problem governed by unsteady diffusion–convection–reaction equation without control constraints is studied. Time discretization is performed by variational discretization using continuous and discontinuous Galerkin methods, while symmetric interior penalty Galerkin with upwinding is used for space discretization. We investigate the commutativity properties of the *optimize-then-discretize* and *discretize-then-optimize* approaches for the continuous and discontinuous Galerkin time discretization. A priori error estimates are derived for fully-discrete state, adjoint and control. The numerical results given for convection dominated problems via *optimize-then-discretize* approach confirm the theoretically observed convergence rates.

1. Introduction

Optimal control problems (OCPs) governed by diffusion–convection–reaction equations arise in environmental control problems, optimal control of fluid flow and in many other applications. It is well known that the standard Galerkin finite element discretization causes non-physical oscillating solutions when convection dominates. Stable and accurate numerical solutions can be achieved by various effective stabilization techniques such as the streamline upwind/Petrov–Galerkin (SUPG) finite element method [1], the local projection stabilization [2], the edge stabilization [3]. Recently, discontinuous Galerkin (dG) methods have gained importance due to their better convergence behaviour, local mass conservation, flexibility in approximating rough solutions on complicated meshes, mesh adaptation and weak imposition of the boundary conditions, see, e.g., [4,5].

In this paper, we solve the OCP governed by diffusion–convection–reaction equation by applying symmetric interior penalty Galerkin (SIPG) method with upwinding in space [6–8] and variational time discretization [9–14].

In recent years, most of the research is concentrated on parabolic OCPs (see for example [15,16]). There are few publications dealing with OCPs governed by non-stationary diffusion–convection–reaction equation. The local DG approximation of the OCP which is discretized by backward Euler in time is studied in [17] and a priori error estimates for semi-discrete OCP is provided in [18]. In [19], the characteristic finite element solution of the OCP is discussed and numerical results are provided. A-priori error estimates for discontinuous Galerkin time discretization for unconstrained parabolic OCPs are proposed

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http://dx.doi.org/10.1016/j.cam.2014.05.002

in [20]. The control-constrained case is investigated in [21] for SIPG method combined with backward Euler discretization. Crank–Nicolson time discretization is applied for OCP of diffusion–convection equation in [22]. To the best of our knowledge, this is the first study on space–time DG discretization of OCPs governed by convection–diffusion–reaction equations.

There exist two different approaches for solving OCPs: *optimize-then-discretize* (*OD*) and *discretize-then-optimize* (*DO*). In the *OD* approach, first the infinite dimensional optimality system is derived containing the state and adjoint equations and the variational inequality. Then, the optimality system is discretized by using a suitable discretization method in space and time. In *DO* approach, the infinite dimensional OCP is discretized and then the finite-dimensional optimality system is derived. The *DO* and *OD* approaches do not commute in general for OCPs governed by diffusion–convection–reaction equation [1]. However, commutativity is achieved in the case of SIPG discretization for steady state problems [4]. For discontinuous Galerkin time discretization, where both trial and test spaces are discontinuous, we show that *OD* and *DO* approaches commute, i.e. the adjoint state is discretized as we do for the state variable. For continuous Galerkin time discretization, where the trial spaces are continuous and the test spaces are discontinuous, *OD* and *DO* approaches do not commute. For a priori error estimates, we use the error analysis in [20] adapted to space–time discontinuous Galerkin discretization. For this purpose, we divide the error analysis in three parts as in [19] using the error estimates for the dG bilinear forms.

The rest of the paper is organized as follows. In Section 2, we define the model problem and then derive the optimality system. In Section 3, we present the symmetric interior penalty Galerkin method with upwinding. In Section 4, we discuss the fully discrete optimality system using variational time discretization methods. In Section 5, we give some auxiliary results, which are needed for the a priori error estimates. In Section 6, we derive convergence estimates for the fully discrete optimality system. In Section 7, the computational details of variational time discretization methods are investigated. In Section 8, numerical results are shown in order to discover the performance of the suggested methods. The paper ends with some conclusions.

2. The optimal control problem

We adopt the standard notations for Sobolev spaces on computational domains and their norms. Let Ω be a bounded convex polygonal domain in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. The inner product in $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$.

We consider the following distributed optimal control problem governed by the unsteady diffusion–convection–reaction equation

$$\underset{u\in L^2(0,T;L^2(\Omega))}{\text{minimize}} \; J(y,u) := \frac{1}{2}\int_0^T \left(\|y-y_d\|^2_{L^2(\Omega)} + \alpha\, \|u\|^2_{L^2(\Omega)} \right) dt, \tag{2.1a}$$

$$\text{subject to } \partial_t y - \epsilon\Delta y + \boldsymbol{\beta}\cdot\nabla y + ry = f + u \quad (x,t)\in\Omega\times(0,T], \tag{2.1b}$$

$$y(x,t) = 0 \quad (x,t)\in\partial\Omega\times[0,T], \tag{2.1c}$$

$$y(x,0) = y_0(x) \quad x\in\Omega. \tag{2.1d}$$

The source function and the desired state are denoted by $f\in L^2(0,T;L^2(\Omega))$ and $y_d\in L^2(0,T;L^2(\Omega))$, respectively. The initial condition is also defined as $y_0(x)\in H^1_0(\Omega)$. The diffusion and reaction coefficients are $\epsilon>0$ and $r\in L^\infty(\Omega)$, respectively. The velocity field $\boldsymbol{\beta}\in(W^{1,\infty}(\Omega))^2$ satisfies the incompressibility condition, i.e. $\nabla\cdot\boldsymbol{\beta}=0$. Furthermore, we assume the existence of the constant C_0 such that $r\geq C_0$ a.e. in Ω so that the well-posedness of the optimal control problem (2.1) is guaranteed. The trial and test spaces are $Y=V=H^1_0(\Omega),\ \forall t\in(0,T]$. The OCP (2.1) problem is written in variational form as follows

$$\underset{u\in L^2(0,T;L^2(\Omega))}{\text{minimize}} \; J(y,u) := \frac{1}{2}\int_0^T \left(\|y-y_d\|^2_{L^2(\Omega)} + \alpha\|u\|^2_{L^2(\Omega)} \right) dt \tag{2.2a}$$

$$\text{subject to } (\partial_t y, v) + a(y,v) = (f+u, v), \quad \forall v\in V,\, t\in I, \tag{2.2b}$$

$$y(x,0) = y_0, \quad x\in\Omega,$$

with

$$a(y,v) = \int_\Omega (\epsilon\nabla y\cdot\nabla v + \boldsymbol{\beta}\cdot\nabla y v + ryv)dx, \qquad (w,v) = \int_\Omega wv dx.$$

It is well known that the functions $(y,u)\in H^1(0,T;L^2(\Omega))\cap L^2(0,T;Y)\times L^2(0,T;L^2(\Omega))$ solve (2.1) if and only if there is an adjoint $p\in H^1(0,T;L^2(\Omega))\cap L^2(0,T;Y)$ such that (y,u,p) is the unique solution of the following optimality system [23],

$$(\partial_t y, v) + a(y,v) = (f+u, v) \quad \forall v\in V,\ y(x,0)=y_0, \tag{2.3a}$$

$$-(\partial_t p, \psi) + a(\psi,p) = -(y-y_d,\psi) \quad \forall\psi\in V,\ p(x,T)=0, \tag{2.3b}$$

$$\int_0^T (\alpha u - p, w-u)dt = 0, \quad \forall w\in L^2(0,T;L^2(\Omega)). \tag{2.3c}$$