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Least squares approximation method for the solution of Volterra–Fredholm integral equations

Qisheng Wang [a,*], Keyan Wang [a], Shaojun Chen [b]

[a] School of Mathematics and Computational Science, Wuyi University, Jiangmen, Guangdong 529020, PR China
[b] Department of Mathematics, Zhoukou Technology and Vocational College, Zhoukou, Henan 466000, PR China

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ABSTRACT

In this paper, an efficient numerical method is developed for solving the Volterra–Fredholm integral equations by least squares approximation method, which is based on a polynomial of degree n to compute an approximation to the solution of Volterra–Fredholm integral equations. The convergence analysis of the approximation solution relative to the exact solution of the integral equation is proved. The reliability and efficiency of the proposed method are demonstrated by some numerical experiments.

1. Introduction

Integral equation has been one of the principal tools in various areas of applied mathematics, physics and engineering, hence, the literature on integral equations and their applications is vast. For example, see [1–4] and the references therein. In this paper, we consider a class of mixed Volterra–Fredholm integral equations of the form

$$A(x)y(x) + B(x)y(h(x)) = f(x) + \lambda_1 \int_a^{h(x)} k_1(x,t)y(t)dt + \lambda_2 \int_a^b k_2(x,t)y(h(t))dt, \tag{1.1}$$

where the functions $k_1(x,t)$, $k_2(x,t)$, $A(x)$, $B(x)$, $h(x)$ and $f(x)$ are known on the interval $[a,b]$, and a, b are constants, $y(x)$ is the continuous function to be determined, $\lambda_i \in R$ $(i = 1, 2)$ and $\lambda_1^2 + \lambda_2^2 \neq 0$. In particular, when $h(x)$ is a first-order polynomial, Eq. (1.1) is reduced to a functional integral equation with proportional delay. As in [5,6], by using the well-known Banach fixed-point theorem, one can easily prove that the solution of (1.1) exists and is unique on $[a,b]$.

Numerical methods for solving integral equations have been studied extensively in the literature, see [7] for details. A squared reminder minimization method for the solution of multi-pantograph equation was first introduced by Bota and Căruntu in [8], and this method was developed in [9]. Best square approximation method was used for solving a mixed linear Volterra–Fredholm integral equation by Chen and Jiang [10]. Similarly, numerical methods such as the Taylor polynomial method was introduced in [11,12], the Taylor collocation method was presented in [13,14] and implementations of such methods for the Volterra–Fredholm integral (integro-differential) equations, nonlinear Schrödinger equation, and high-order linear pantograph equations, respectively.

* Corresponding author. Tel.: +86 0750 3296583; fax: +86 0750 3296583.
E-mail addresses: wqs9988@yahoo.cn, wqs9988@aliyun.com (Q. Wang).

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In this study, we are concerned with the application of least squares approximation method to the numerical solution of Volterra–Fredholm integral equations, which is based on a polynomial of degree n to compute an approximation to the solution of Volterra–Fredholm integral equations. The basic ideas of the previous works [8–10] are developed and applied to (1.1). Also, we will compare the error of the approximate solution with the exact solution of the Volterra–Fredholm integral equations by our method and those methods in [4,11–14].

2. Method of solution

Throughout of this paper, we always assume that the functions $A(x)$, $B(x)$ and k_i $(i = 1, 2)$ satisfy some conditions such that the solution of (1.1) exists and is unique. Now we study the least squares approximation method to approximate the solution of Eq. (1.1). Firstly, we define the operator:

$$T(x, y(x)) = A(x)y(x) + B(x)y(h(x)) - f(x) - \lambda_1 \int_a^{h(x)} k_1(x, t)y(t)dt - \lambda_2 \int_a^b k_2(x, t)y(h(t))dt. \tag{2.1}$$

For positive integer $n > 0$, suppose $\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x)$ are linearly independent functions on the interval $[a, b]$, $\Phi_n = span\{\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x)\}$ is generated by the linear space. Let $y_n(x) \in \Phi_n$, there exist numbers $c_0, c_1, \ldots, c_n$ such that

$$y_n(x) = \sum_{i=0}^{n} c_i\varphi_i(x). \tag{2.2}$$

Substituting (2.2) into Eq. (1.1), we can obtain the expression:

$$\begin{aligned} T(x, y_n(x)) &= A(x)y_n(x) + B(x)y_n(h(x)) - f(x) - \lambda_1 \int_a^{h(x)} k_1(x, t)y_n(t)dt - \lambda_2 \int_a^b k_2(x, t)y_n(h(t))dt \\ &= \sum_{i=0}^{n} c_i \cdot \left[A(x)\varphi_i(x) + B(x)\varphi_i(h(x)) - \lambda_1 \int_a^{h(x)} k_1(x, t)\varphi_i(t)dt - \lambda_2 \int_a^b k_2(x, t)\varphi_i(h(t))dt\right] - f(x) \\ &= \sum_{i=0}^{n} c_i \cdot \alpha_i(x) - f(x), \end{aligned} \tag{2.3}$$

where $\alpha_i(x) = A(x)\varphi_i(x) + B(x)\varphi_i(h(x)) - \lambda_1 \int_a^{h(x)} k_1(x, t)\varphi_i(t)dt - \lambda_2 \int_a^b k_2(x, t)\varphi_i(h(t))dt, i = 0, 1, \ldots, n.$

For any $x \in [a, b]$, $R_n(x) = T(x, y_n(x)) - T(x, y(x))$ is called the n-order remaining items of Eq. (1.1), where

$$\begin{aligned} R_n(x) &= A(x)(y_n(x) - y(x)) + B(x)(y_n(h(x)) - y(h(x))) - \lambda_1 \int_a^{h(x)} k_1(x, t)(y_n(t) - y(t))dt \\ &\quad - \lambda_2 \int_a^b k_2(x, t)(y_n(h(t)) - y(h(t)))dt. \end{aligned}$$

Remark 2.1. If $R_n(x) = 0$, then $y(x) = y_n(x)$; if $\lim_{n\to\infty} R_n(x) = 0$, then $\lim_{n\to\infty} y_n(x) = y(x)$.

Remark 2.2. For any $x \in [a, b]$, if $R_n(x) \equiv 0$, then $y_n(x)$ is an exact solution of Eq. (1.1); if $\lim_{n\to\infty} R_n(x) = 0$, then $y_n(x)$ converges to the exact solution of Eq. (1.1).

In the following, let

$$I = I(c_0, c_1, \ldots, c_n) = \int_a^b T^2(x, y_n(x))dx. \tag{2.4}$$

The problem is to find real coefficients $c_0, c_1, \ldots, c_n$ that will minimize I. A necessary condition for the numbers $c_0, c_1, \ldots, c_n$ to minimize I is that

$$\frac{\partial I}{\partial c_i} = 0,$$

for each $i = 0, 1, \ldots, n$. By the relation (2.4), we can easily get

$$\begin{aligned} \frac{\partial I}{\partial c_i} &= 2\int_a^b T(x, y_n(x))\frac{\partial T(x, y_n(x))}{\partial c_i}dx \\ &= 2\int_a^b \left\{\sum_{j=0}^{n}\left[A(x)\varphi_j(x) + B(x)\varphi_j(h(x)) - \lambda_1 \int_a^{h(x)} k_1(x, t)\varphi_j(t)dt - \lambda_2 \int_a^b k_2(x, t)\varphi_j(h(t))dt\right] \cdot c_j - f(x)\right\} \end{aligned}$$