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A uniform monotone alternating direction scheme for nonlinear singularly perturbed parabolic problems

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A B S T R A C T

This paper deals with a monotone alternating direction (ADI) scheme for solving nonlinear singularly perturbed parabolic problems. Monotone sequences, based on the method of upper and lower solutions, are constructed for a nonlinear difference scheme which approximates the nonlinear parabolic problem. The monotone sequences possess quadratic convergence rate. An analysis of uniform convergence of the monotone ADI scheme to the solutions of the nonlinear difference scheme and to the continuous problem is given. Numerical experiments are presented.

1. Introduction

In this paper we give a numerical treatment for the nonlinear singularly perturbed parabolic problem in the form

$$\begin{aligned} &u_t - Lu + f(x,y,t,u) = 0, \quad Lu \equiv \mu^2(u_{xx}+u_{yy}), \\ &(x,y,t) \in Q = \omega \times (0,T], \quad \omega = \{0<x<1\} \times \{0<y<1\}, \\ &u(x,y,t) = 0, \quad (x,y,t) \in \partial\omega \times (0,T], \\ &u(x,y,0) = \psi(x,y), \quad (x,y) \in \overline{\omega}, \end{aligned} \tag{1}$$

where μ is a small positive parameter, $\partial\omega$ is the boundary of ω, the functions f and ψ are smooth in their respective domains, and f satisfies the constraint

$$f_u \geq \beta, \quad (x,y,t,u) \in \overline{\omega} \times [0,T] \times (-\infty,\infty), \tag{2}$$

where $\beta = \text{const} > 0$. This assumption can always be obtained via a change of variables. Indeed, introduce $z(x,y,t) = e^{-\lambda t}u(x,y,t)$, where λ is a constant. Now, $z(x,y,t)$ satisfies (1) with $\varphi = \lambda z + e^{-\lambda t} f(x,y,t,e^{\lambda t} z)$, instead of f, and we have $\varphi_z = \lambda + f_u$. Thus, if $\lambda \geq -\min f_u + \beta$, where minimum is taking over the domain from (2), we conclude $\varphi_z \geq \beta$.

For $\mu \ll 1$, the problem is singularly perturbed and characterized by boundary layers (regions with rapid change of solutions) near boundary $\partial\omega$ (see [1] for details). Various reaction–diffusion-type problems in chemical, physical and engineering sciences are described by problem (1).

In the study of numerical methods for nonlinear singularly perturbed problems, the two major points to be developed are: (i) constructing robust difference schemes (this means that unlike classical schemes, the error does not increase to infinity, but rather remains bounded, as the small parameter approaches zero); (ii) obtaining reliable and efficient computing algorithms for solving nonlinear discrete problems.

We shall employ a two-time level implicit scheme for approximating the semilinear problem (1). Alternating direction implicit (ADI) methods are very efficient methods for solving two or three dimensional parabolic problems. At each

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time-step, the ADI method reduces two or three dimensional problems to a succession of one dimensional problems, and, usually, one needs only to solve a sequence of tridiagonal systems. In the case of the nonlinear reaction function f in (1), the corresponding discrete problems become systems of nonlinear algebraic equations.

A fruitful method for solving the nonlinear difference scheme is the method of upper and lower solutions and its associated monotone iterations. By using upper and lower solutions as two initial iterations, one can construct two monotone sequences which converge monotonically from above and below, respectively, to a solution of the problem. The above monotone iterative method is well known and has been widely used for continuous and discrete elliptic and parabolic boundary value problems. Most of publications on this topic involve monotone iterative schemes whose rate of convergence is linear. Accelerated monotone iterative methods for solving discrete parabolic problems are presented in [2–4]. An advantage of this accelerated approach is that it leads to sequences which converge quadratically.

In [5,6], the ADI method based on the Douglas–Rachford ADI scheme [7] is applied to linear singularly perturbed reaction–diffusion problems of type (1). This ADI method is shown to be uniformly convergent (robust) with respect to the small parameter μ on special nonuniform meshes.

In this paper, we construct a nonlinear ADI scheme based on a modification of the Douglas–Rachford ADI scheme [7]. A monotone iterative method with quadratic convergence rate from [2] is in use for solving nonlinear discrete systems. We consider the case when on each time level a nonlinear difference scheme is solved inexactly, and give an analysis of convergence of a monotone ADI scheme on the whole interval of integration $[0, T]$.

The structure of the paper as follows. In Section 2, we introduce a nonlinear difference scheme for the numerical solution of (1), (2). In Section 3, we construct a nonlinear ADI scheme. The new monotone ADI scheme is presented in Section 4. Monotone properties of the ADI scheme are established. Based on these properties, existence and uniqueness of the solution to the nonlinear ADI scheme are proved. In Section 5, we show that on each time level the monotone iterative method possesses quadratic convergence rate. We analyze a convergence rate of the monotone ADI scheme on the whole interval of integration $[0, T]$. Section 6 deals with uniform convergence of the monotone ADI scheme to the nonlinear parabolic problem (1), (2). Section 7 presents results of numerical experiments.

2. The nonlinear difference scheme

On $\overline{Q}$ introduce a rectangular mesh $\overline{\omega}^h \times \overline{\omega}^\tau$, $\overline{\omega}^h = \overline{\omega}^{hx} \times \overline{\omega}^{hy}$:

$$\begin{aligned}
&\overline{\omega}^{hx} = \left\{x_i,\ 0 \le i \le N_x;\ x_0 = 0,\ x_{N_x} = 1;\ h_{xi} = x_{i+1} - x_i\right\},\\
&\overline{\omega}^{hy} = \left\{y_j,\ 0 \le j \le N_y;\ y_0 = 0,\ y_{N_y} = 1;\ h_{yj} = y_{j+1} - y_j\right\},\\
&\overline{\omega}^{\tau} = \left\{t_k, 0 \le k \le N_\tau;\ t_0 = 0, t_{N_\tau} = T;\ \tau_k = t_k - t_{k-1}\right\}.
\end{aligned} \tag{3}$$

For solving (1), consider the nonlinear implicit difference scheme

$$\mathcal{L}U(p, t_k) + f(p, t_k, U) - \tau_k^{-1} U(p, t_{k-1}) = 0, \quad (p, t_k) \in \omega^h \times (\overline{\omega}^\tau \setminus \{0\}), \tag{4}$$

with the boundary and initial conditions

$$\begin{aligned}
&U(p, t_k) = 0, \quad (p, t_k) \in \partial\omega^h \times (\overline{\omega}^\tau \setminus \{0\}),\\
&U(p, 0) = \psi(p), \quad p \in \overline{\omega}^h,
\end{aligned}$$

where $\partial\omega^h$ is the boundary of $\overline{\omega}^h$. When no confusion arises, we write $f(p, t_k, U(p, t_k)) = f(p, t_k, U)$. The difference operator $\mathcal{L}$ is defined by

$$\begin{aligned}
&\mathcal{L}U(p, t_k) = \mathcal{L}^h U(p, t_k) + \tau_k^{-1} U(p, t_k),\\
&\mathcal{L}^h U = \mathcal{L}_x^h U + \mathcal{L}_y^h U, \qquad \mathcal{L}_\nu^h U = -\mu^2 \mathcal{D}_\nu^2 U, \quad \nu = x, y,
\end{aligned}$$

where $\mathcal{D}_x^2 U$ and $\mathcal{D}_y^2 U$ are the central difference approximations to the second derivatives

$$\begin{aligned}
&\mathcal{D}_x^2 U_{ij}^k = (\hbar_{xi})^{-1} \left[\left(U_{i+1,j}^k - U_{ij}^k\right)(h_{xi})^{-1} - \left(U_{ij}^k - U_{i-1,j}^k\right)\left(h_{x,i-1}\right)^{-1}\right],\\
&\mathcal{D}_y^2 U_{ij}^k = \left(\hbar_{yj}\right)^{-1} \left[\left(U_{i,j+1}^k - U_{ij}^k\right)\left(h_{yj}\right)^{-1} - \left(U_{ij}^k - U_{i,j-1}^k\right)\left(h_{y,j-1}\right)^{-1}\right],\\
&\hbar_{xi} = 2^{-1}\left(h_{x,i-1} + h_{xi}\right), \qquad \hbar_{yj} = 2^{-1}\left(h_{y,j-1} + h_{yj}\right), \qquad U_{ij}^k \equiv U\left(x_i, y_j, t_k\right).
\end{aligned}$$

On each time level t_k, $k \ge 1$, introduce the linear difference problem

$$\begin{aligned}
&(\mathcal{L} + c(p, t_k)I)W(p, t_k) = \Phi(p, t_k), \quad p \in \omega^h,\\
&W(p, t_k) = g(p, t_k), \quad p \in \partial\omega^h,
\end{aligned} \tag{5}$$

where I is the identity operator. We are concerned with maximal nodal errors, so we use the norm

$$\|W(\cdot, t_k)\|_{\overline{\omega}^h} = \max_{p \in \overline{\omega}^h} |W(p, t_k)|.$$

We now formulate the maximum principle and give an estimate to the solution of (5).