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Discrepancy norm: Approximation and variations

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ABSTRACT

This paper introduces an approach for the minimization of the discrepancy norm. The general idea is to replace the infinity norms appearing in the definition by L^p norms which are differentiable and to make use of this approximation for local optimization.

We will show that the discrepancy norm can be approximated up to any ε and the robustness of this approximation is shown. Moreover, analytical formulation of the derivative of the discrepancy correlation function is given.

In a following step we extend the results to higher dimensional data and derive the related forms for approximations and differentiations.

1. Introduction

This paper deals with the use of the discrepancy norm as introduced by Weyl [1] in the context of signal analysis. A great deal of research in discrepancy theory has been done in numerical integration [2], analysis of randomness [3] and uniformity [4], low-discrepancy sequences for sampling [5], computational geometry [6], and geometric approximation [7]. We invite the reader to have a look at the book of Chazelle for a rather complete review of topics in discrepancy theory [8].

Here we investigate another application of discrepancy theory, namely to signal analysis. As re-introduced by Neunzert and Welton [9] and later by Moser [10] the discrepancy norm becomes an interesting tool for pattern recognition. We revisit here some previous work [11] where an approximation of the discrepancy norm was developed for discrete time series analysis. We extend our results to continuous measurable functions and compute its analytic derivative that can be used for optimization purposes.

In a second step we generalize these results to higher-dimensional signals by considering directional discrepancies and combining them together. Artificial experiments are provided to validate our theoretical findings.

Our contribution is two-fold. We first extend previous results [11] obtained for one-dimensional discrete signals to the more general case of continuous measurable functions. We show that our approximation can be done up to any ε for any function from a measurable space with finite measure. These results are also proven in higher dimension. Second we compute the derivative of this approximation that comes handy for optimization (for instance in the context of signal alignment).

We first start by reviewing the basics regarding the discrepancy norm in Section 2: its definition and its useful properties. Of particular interest for signal alignment is the monotonicity property of its autocorrelation function (Property C3). Then Section 3 introduces the approximation through L^p-norms and its derivative for one dimensional data. We show some convergence criteria to prove the robustness of the approaches. Section 4 extends the previous results to higher dimensional

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data. We see how the problem arise in the discrepancy norm's definition when dealing with higher dimensions. Numerical experiments are given along the theoretical results.

2. Discrepancy norm

2.1. Introduction and first definitions

Discrepancy measurements have been studied for a very long time and date back to Hermann Weyl's theory [1]. Here we are more interested in its application in terms of similarity measure [12].

As seen in [13,14], the discrepancy norm introduced here is particularly suited for image registration. Other approaches have been studied recently for pattern recognition [9] and vision purposes [15]. It has been reintroduced for general measurable functions by Moser [10].

Definition 1 (*Discrepancy Norm*). Let $(\mathbb{R}, \Sigma, \mu)$ be a measure space with μ a finite measure. The discrepancy norm of a function $f \in L(\mathbb{R}, \mu)$ is defined as

$$\|f\|_D = \sup_{[a,b]\subset\mathbb{R}} \left| \int_a^b f \, \mathrm{d}\mu \right| .$$

2.2. Properties

It has been proven that the discrepancy norm shows particularly interesting properties for analyzing misalignments in signals [12] compared to other traditional measures.

Definition 2 (*Misalignment Functions*). Given a distance measure

$$d : L(\mathbb{R}, \mu) \times L(\mathbb{R}, \mu) \to \mathbb{R}^+$$

one can define a misalignment measure of a signal as follows:

$$\Delta_d[f](t) := d(f, f_t)$$

where f_t is a translation of f defined as $\forall x \in \mathbb{R}, f_t(x) = f(x - t)$.

In [10] three criteria have been introduced to evaluate the ability of a metric to act as a misalignment measure:

C1 Positive-definiteness:

$$\Delta_d[f](t) = 0 \Leftrightarrow t = 0$$

C2 Continuity:

$$\Delta_d[f](t) \to 0, \quad \text{for } t \to 0$$

C3 Monotonicity:

$$\forall 1 \le \lambda, \quad \forall t \in \mathbb{R}, \qquad \Delta_d[f](t) \le \Delta_d[f](\lambda t).$$

Property 1 (*Compatibility of the Discrepancy Norm*). *Given any measurable function $f \in L(\mathbb{R}, \mu)$, $f \ge 0$, μ almost everywhere, the discrepancy norm fulfills the three criteria described above.*

Note that this result is stated on the real line but it could be adapted on the torus to deal with periodic functions. With such function, criteria C1 and C3 need to be adapted. We actually have a more stronger result regarding the continuity criterion:

Property 2 (*Lipschitz Continuity*). *The variations of the discrepancy misalignment function are bounded by its infinity norm:*

$$\Delta_{\|\cdot\|_D}[f](t) \le |t| \|f\|_\infty .$$

The next property defines another computation for the discrepancy norm which appears to be much faster in the applications.

Property 3 (*Linear Computation*). *Let $f \in L(\mathbb{R}, \mu)$, the following holds true*

$$\|f\|_D = \max_{b\in\mathbb{R}} \int_{-\infty}^b f \, \mathrm{d}\mu - \min_{a\in\mathbb{R}} \int_{-\infty}^a f \, \mathrm{d}\mu .$$

This last formulation allows us to compute the discrepancy of a function by means of integral images [16,17]. These integral images are, in practice, computed in a linear time with respect to the number of samples.