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Stability estimate and the modified regularization method for a Cauchy problem of the fractional diffusion equation

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ABSTRACT

In this paper we investigate a non-characteristic Cauchy problem for a fractional diffusion equation. Using the Fourier transformation technique, we give a conditional stability estimate on the solution. Since the problem is highly ill-posed in the Hadamard sense, a modified version of the Tikhonov regularization technique is devised for stable numerical reconstruction of the solution. An error bound with optimal order is proven. For illustration, several numerical experiments are constructed to demonstrate the feasibility and efficiency of the proposed method.

1. Introduction

Partial differential equations with fractional order arose from the studies of continuous random walk and Lévy motion [1–4] and high-frequency financial data [5,6]. Among these studies the modeling of advection and dispersion phenomena in groundwater hydrology to simulate the transport of passive tracers carried by fluid flow in a porous medium resulted in a partial differential equation with fractional order [7–11]. For economical and safe control of plants, there is a great demand on a good estimation of the concentration and flow of pollutants from only spatially observed data during advection and dispersion processes. In general, fluid flow and diffusion phenomena are governed by a fractional advection–dispersion equation. If the initial concentration distribution and boundary conditions are given, a complete recovery of the unknown solution is attainable from solving a well-posed forward problem [12,13]. In real-life, however, the boundary conditions can be missed and the distribution data can only be collected at a particular time. This is usually referred to an ill-posed backward determination problem, which is in nature "unstable" because the unknown solution and its derivatives have to be determined from indirect observable data which contain measurement error. The major difficulty in establishing any numerical algorithm for approximating the solution is due to the severe ill-posedness of the problem and the ill-conditioning of the resultant discretized matrix.

Anomalous diffusion has been found in a broad variety of physics and engineering disciplines, for example, electron transportation [14], dissipation [15], and heat conduction [16]. It is well known that the continuous-time random walk is a microscopic model for anomalous diffusion. By an argument similar to the derivation of the classical diffusion equation from the random walk model, one can derive fractional diffusion models [17,18]. In recent years, the study of the fractional derivative anomalous diffusion equation has attracted attention from many researchers.

In practical real-life problems, diffusion is anomalous but the boundary data can only be measured on a part of the boundary. This leads to the Cauchy problem of the fractional diffusion equation, which is by nature ill-posed, refer to [19]

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for more details. To the knowledge of the authors, there is still a lack of full understanding on inverse problems for fractional differential equations. Recent works can be found from [20–25]. In this paper we aim at investigating the conditional stability and a modified regularization method for a Cauchy problem of the fractional diffusion equation.

Cauchy problems of the classical diffusion equation arise from many physical and engineering disciplines. It is well known that the Cauchy problem of the heat diffusion equation is ill-posed in the Hadamard sense that a small measurement error in the boundary data can induce an enormous error in the solution. Under an additional condition, a continuous dependence of the solution on the Cauchy data can be obtained. This is called conditional stability [26]. Due to the severe ill-posedness of the problem, it is difficult to solve the Cauchy problem of the heat diffusion equation by using classical numerical methods without using some kinds of regularization strategies [27]. Theoretical concepts and computational implementation related to the Cauchy problem of the heat diffusion equation have been discussed by many authors. Refer to, for instances, our recent works on [28,29] for computational aspects and [30] for theoretical aspects.

The paper is organized as follows. In Section 2, we formulate a Cauchy problem for the fractional diffusion equation, then give the conditional stability estimate on the solution in Section 3 by using the Fourier transformation technique. For stable reconstruction of the highly ill-posed solution, we adapt the usage of a modified Tikhonov regularization technique with theoretical regularization error estimate in Section 4. In the last section, several numerical examples are constructed to demonstrate that the proposed method is of feasibility and efficiency.

2. Mathematical formulation of the Cauchy problem

Consider the following two Cauchy problems for the fractional diffusion equation:

Problem I.

$$\begin{aligned}
&\frac{\partial^\beta u}{\partial t^\beta} - u_{xx} = 0, \quad 0 < x < a,\ t > 0,\\
&u(0,t) = \phi(t), \quad t > 0,\\
&u_x(0,t) = 0, \quad t > 0,\\
&u(x,0) = 0, \quad 0 < x < a.
\end{aligned} \tag{2.1}$$

Problem II.

$$\begin{aligned}
&\frac{\partial^\beta v}{\partial t^\beta} - v_{xx} = 0, \quad 0 < x < a,\ t > 0,\\
&v(0,t) = 0, \quad t > 0,\\
&v_x(0,t) = h(t), \quad t > 0,\\
&v(x,0) = 0, \quad 0 < x < a,
\end{aligned} \tag{2.2}$$

where the time fractional derivative $\frac{\partial^\beta u}{\partial t^\beta}$ is the Caputo fractional derivative of order β $(0 < \beta \le 1)$ defined in [31]

$$\frac{\partial^\beta u}{\partial t^\beta} = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\beta}, \quad 0 < \beta < 1, \tag{2.3}$$

$$\frac{\partial^\beta u}{\partial t^\beta} = \frac{\partial u(x,t)}{\partial t}, \quad \beta = 1. \tag{2.4}$$

The Cauchy problem is then to recover the unknown solutions $u(x,t)$ and $v(x,t)$ from the given data $\phi(t)$, $h(t)$, respectively. In reality, the measurement data ϕ, h contain noises and hence the solutions have to be sought from the data functions $\phi^\delta(t), h^\delta(t) \in L^2(\mathbb{R})$ satisfying

$$\|\phi^\delta(t) - \phi(t)\| + \|h^\delta(t) - h(t)\| \le \delta, \tag{2.5}$$

where the level of tolerance $\delta > 0$ represents a bound on the measurement error and $\|\cdot\|$ denotes the L^2-norm. Assume that there exists a constant $E > 0$ such that the following a-priori bounds exist:

$$\max\{\|u(a,\cdot)\|,\ \|v(a,\cdot)\|\} \le E, \tag{2.6}$$

and let $w(x,t) = u(x,t) + v(x,t)$. Consider the problem

Problem III.

$$\begin{aligned}
&\frac{\partial^\beta w}{\partial t^\beta} - w_{xx} = 0, \quad 0 < x < a,\ t > 0,\\
&w(0,t) = \phi(t), \quad t > 0,\\
&w_x(0,t) = h(t), \quad t > 0,\\
&w(x,0) = 0, \quad 0 < x < a.
\end{aligned} \tag{2.7}$$