Journal of Computational and Applied Mathematics 272 (2014) 251–257

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Explicit formulae for computing Euler polynomials in terms of Stirling numbers of the second kind

Bai-Ni Guo [a,*], Feng Qi [b,c]

[a] School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

[b] College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, 028043, China

[c] Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300387, China

ARTICLE INFO

Article history:
Received 20 November 2013
Received in revised form 7 April 2014

MSC:
11B68
11B73
33B10

Keywords:
Explicit formula
Euler polynomial
Two-parameter Euler polynomial
Stirling number of the second kind
Identity
Exponential function

ABSTRACT

In the paper, the authors elementarily unify and generalize eight identities involving the functions $\frac{\pm 1}{e^{\pm t}-1}$ and their derivatives. By one of these identities, the authors establish two explicit formulae for computing Euler polynomials and two-parameter Euler polynomials, which are a newly introduced notion, in terms of Stirling numbers of the second kind.

1. Introduction

In [1], the following eight identities were elementarily and inductively established.

Theorem 1.1. *([1, Theorems 2.1–2.4 and Corollaries 2.1–2.4]) For $k \in \mathbb{N}$, we have*

$$\left(\frac{1}{e^t-1}\right)^{(k)} = \sum_{m=1}^{k+1} \lambda_{k,m} \left(\frac{1}{e^t-1}\right)^m, \qquad \left(\frac{1}{1-e^{-t}}\right)^{(k)} = \sum_{m=1}^{k+1} \mu_{k,m} \left(\frac{1}{1-e^{-t}}\right)^m, \tag{1.1}$$

$$\left(\frac{1}{1-e^{-t}}\right)^{(k)} = \sum_{m=1}^{k+1} \lambda_{k,m} \left(\frac{1}{e^t-1}\right)^m, \qquad \left(\frac{1}{e^t-1}\right)^{(k)} = \sum_{m=1}^{k+1} \mu_{k,m} \left(\frac{1}{1-e^{-t}}\right)^m, \tag{1.2}$$

$$\left(\frac{1}{1-e^{-t}}\right)^k = \sum_{m=1}^{k} a_{k,m-1} \left(\frac{1}{1-e^{-t}}\right)^{(m-1)}, \qquad \left(\frac{1}{e^t-1}\right)^k = \sum_{m=1}^{k} b_{k,m-1} \left(\frac{1}{e^t-1}\right)^{(m-1)}, \tag{1.3}$$

* Corresponding author. Tel.: +86 22 83956502; fax: +86 391 3980618.
E-mail addresses: bai.ni.guo@gmail.com, bai.ni.guo@hotmail.com (B.-N. Guo), qifeng618@qq.com, qifeng618@gmail.com, qifeng618@hotmail.com (F. Qi).
URLs: http://www.researcherid.com/rid/C-8032-2013 (B.-N. Guo), http://qifeng618.wordpress.com (F. Qi).

http://dx.doi.org/10.1016/j.cam.2014.05.018
0377-0427/

$$\left(\frac{1}{1-e^{-t}}\right)^k = 1+\sum_{m=1}^{k} a_{k,m-1}\left(\frac{1}{e^t-1}\right)^{(m-1)}, \qquad \left(\frac{1}{e^t-1}\right)^k = 1+\sum_{m=1}^{k} b_{k,m-1}\left(\frac{1}{1-e^{-t}}\right)^{(m-1)}, \tag{1.4}$$

where

$$\lambda_{k,m} = (-1)^k (m-1)! S(k+1,m), \qquad \mu_{k,m} = (-1)^{m-1}(m-1)! S(k+1,m), \tag{1.5}$$

$$a_{k,m-1} = (-1)^{m^2+1} M_{k-m+1}(k,m), \qquad b_{k,m-1} = (-1)^{k-m} a_{k,m-1}, \tag{1.6}$$

$$M_j(k,i) = \begin{vmatrix} \frac{1}{(i-1)!}\binom{k}{i} & S(i+1,i) & S(i+2,i) & \cdots & S(i+j-1,i) \\ \frac{1}{i!}\binom{k}{i+1} & S(i+1,i+1) & S(i+2,i+1) & \cdots & S(i+j-1,i+1) \\ \frac{1}{(i+1)!}\binom{k}{i+2} & 0 & S(i+2,i+2) & \cdots & S(i+j-1,i+2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(i+j-2)!}\binom{k}{i+j-1} & 0 & 0 & \cdots & S(i+j-1,i+j-1) \end{vmatrix}, \quad j \in \mathbb{N}, \tag{1.7}$$

and

$$S(k,m) = \frac{1}{m!}\sum_{\ell=1}^{m} (-1)^{m-\ell}\binom{m}{\ell}\ell^k, \quad 1 \le m \le k \tag{1.8}$$

are Stirling numbers of the second kind which may be generated by

$$\frac{(e^x-1)^k}{k!} = \sum_{n=k}^{\infty} S(n,k)\frac{x^n}{n!}, \quad k \in \mathbb{N}. \tag{1.9}$$

It was pointed out in [1, Remark 5.3] that the above eight identities involving the functions $\frac{\pm 1}{e^{\pm t}-1}$ and their derivatives are equivalent to each other.

By virtue of the first identity in (1.1), among other things, an explicit formula for computing Bernoulli numbers B_{2k}, which are defined by the power series expansion

$$\frac{t}{e^t-1} = \sum_{i=0}^{\infty} B_i \frac{t^i}{i!} = 1 - \frac{t}{2} + \sum_{k=1}^{\infty} B_{2k}\frac{t^{2k}}{(2k)!} \tag{1.10}$$

for $|t| < 2\pi$, in terms of Stirling numbers of the second kind $S(n,k)$, was discovered in [1] as follows.

Theorem 1.2 (*[1, Theorem 3.1]*). *For $k \in \mathbb{N}$, Bernoulli numbers B_{2k} may be computed by*

$$B_{2k} = 1 + \sum_{m=1}^{2k-1} \frac{S(2k+1,m+1)S(2k,2k-m)}{\binom{2k}{m}} - \frac{2k}{2k+1}\sum_{m=1}^{2k}\frac{S(2k,m)S(2k+1,2k-m+1)}{\binom{2k}{m-1}}. \tag{1.11}$$

In [2], making use of Faà di Bruno's formula, combinatorial techniques, and much knowledge on Bell polynomials of the second kind and Stirling numbers of the first and second kinds, the above eight identities were generalized and unified as follows.

Theorem 1.3 (*[2, Theorems 3.1 and 3.2]*). *For $\alpha, \lambda \in \mathbb{R}$,*

(1) *when $n \in \mathbb{N}$, we have*

$$\left(\frac{1}{1-\lambda e^{\alpha t}}\right)^{(n)} = (-1)^n \alpha^n \sum_{k=1}^{n+1} (-1)^{k-1}(k-1)! S(n+1,k)\left(\frac{1}{1-\lambda e^{\alpha t}}\right)^k; \tag{1.12}$$