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On the simultaneous refinement of the zeros of H-palindromic polynomials

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ABSTRACT

In this paper we propose a variation of the Ehrlich–Aberth method for the simultaneous refinement of the zeros of H-palindromic polynomials.

1. Introduction

The design of efficient numerical methods for solving structured generalized eigenvalue problems has attracted a growing interest in recent years due to application demands. Some interesting examples have been included in the MATLAB toolbox NLEVP [1]. In this paper we are specifically concerned with polynomial H-palindromic eigenvalue problems of the form

$$P(\lambda)\boldsymbol{x} = \left(\sum_{i=0}^{k} A_i\lambda^i\right)\boldsymbol{x} = \boldsymbol{0}, \qquad A_{k-i}^H = A_i \in \mathbb{C}^{n\times n}, \quad i = 0, \ldots, k. \tag{1}$$

The structure in the coefficient matrices of (1) induces symmetries in the spectrum of the matrix polynomial. If λ is an eigenvalue then $1/\bar{\lambda}$ is also an eigenvalue and this pairing holds even for the zero eigenvalue, its counterpart being an infinite eigenvalue.

Some variants of the explicit and implicit QR eigenvalue algorithm have been devised for dealing with H-palindromic eigenvalue problems [2]. Since the corresponding structured Schur forms exist under additional conditions these algorithms are restricted to certain subclasses. Some methods for computing a structured Schur form from an unstructured one have been also proposed which can be used to post-process the output of the customary QR and QZ algorithms [3]. However these refinement techniques are subjected to the same restrictions and can suffer from numerical difficulties near exceptional eigenvalue configurations.

This paper is concerned with the computation of a structured approximation of the spectrum of a polynomial H-palindromic eigenvalue problem by means of a root-finding method. Our contribution is much in the spirit of the refinement techniques proposed in [3]. The approach taken here consists of finding the structured approximation by using a zero-finding

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algorithm applied for the refinement of an unstructured approximation providing a set of initial guesses. Since the focus is on the computation of the spectrum rather than of the Schur form, our approach can virtually circumvent restrictions due to the occurrence of exceptional eigenvalues.

Structure preserving root-finders have been proposed in [4] and [5] for dealing with real and T-palindromic polynomials, respectively. Both algorithms rely upon the computation of quadratic factors associated with the desired pairing of the zeros. These factors are simultaneously approximated by using some modification of the Ehrlich–Aberth process [6,7]. The method presented in [4] makes use of Bairstow scheme for refining the coefficients of the quadratic factor. The scheme reduces to the Newton–Raphson iteration applied to the nonlinear system defined from the coefficients in the remainder generated from the synthetic division algorithm applied to the polynomial and its approximated quadratic factor.

The goal of this paper is to devise a similar strategy for covering with H-palindromic polynomials. The derivation of the resulting algorithm is treated in Section 2. First we observe that up to a suitable normalization a quadratic H-palindromic polynomial can be determined by two real parameters. Then we exploit the properties of a certain polynomial Diophantine equation involving the given H-palindromic polynomial of degree n and two approximated factors of degree $n-2$, and 2, respectively. It is shown that under some mild assumptions the solution consists of a real T-palindromic polynomial of degree 2. In this way by applying the Newton–Raphson iteration to the nonlinear system given from setting the coefficients of this polynomial equal to zero we obtain a method for the refinement of the quadratic factor. Under some specific circumstances the eigenvalue pairing $(\lambda, 1/\bar{\lambda})$ reduces to $(\lambda, 1/\lambda)$ and, therefore, the resulting method can also be used to solve certain T-palindromic eigenvalue problems. However, it should be noticed that up to a suitable normalization a quadratic T-palindromic polynomial can be determined by one single complex parameter. In this way the exploitation of the T-palindromic spectral symmetry by directly using the specialized method proposed in [5] generally achieves better cost and convergence properties.

Results of numerical experiments to test our algorithm are given in Section 3. Several examples of quadratic eigenvalue problems with spectral symmetry $(\lambda, 1/\bar{\lambda})$ are considered. An initial structured approximation of the spectrum is generated from the output returned by the `polyeig` function in MATLAB – or the improved option `quadeig` [8] for quadratic eigenvalue problems – and then refined by applying our simultaneous refinement procedure. Numerical tests indicate that the proposed approach is numerically robust and computationally efficient.

2. The derivation of the algorithm

A nonzero polynomial $p(z) \in \mathbb{C}[z]$ of degree n is H-palindromic if for a certain integer k, with $0 \le k \le n$, it holds $p(z) = z^k q(z)$, $q(0) \neq 0$ and $\bar{q}(z) = z^{n-k} q(1/z)$, where $\bar{q}(z)$ is the polynomial obtained from $q(z)$ by conjugating its coefficients. Observe that k is uniquely determined as the algebraic multiplicity of 0 as a root of $p(z) = 0$. If $n-k$ is even then the nonzero roots come in pairs $(\lambda_j, 1/\bar{\lambda}_j)$, $1 \le j \le (n-k)/2$. This pairing can be extended to zero roots by introducing k additional zeros at infinity and then we say that the polynomial $p(z)$ has $n_p = n+k$ zeros including zeros at infinity. Similarly, if the conditions $p(z) = z^k q(z)$, $q(0) \neq 0$ and $q(z) = z^{n-k} q(1/z)$ are satisfied then $p(z)$ is said to be T-palindromic and $n_p \ge n$ denotes the total number of its zeros including possible zeros at infinity.

A palindromic polynomial admits an irreducible factorization in terms of palindromic factors. Specifically, let $p(z) \in \mathbb{C}[z]$ be a H-palindromic polynomial with an even number n_p of zeros including zeros at infinity. Then it can be factored in the form [9]

$$p(z) = cz^k \prod_{j=1}^{2m} (a_j + \bar{a}_j z) \prod_{j=1}^{\ell} (z + b_j)(\bar{b}_j z + 1),$$

where $c \in \mathbb{R} \setminus \{0\}$, $a_j, b_j \in \mathbb{C}$, $|a_j| = 1$ and $|b_j| \neq 1$. This factorization can be rewritten into a more compact way by grouping the zeros at the origin with their reciprocals at infinity

$$p(z) = c \prod_{j=1}^{2m} (a_j + \bar{a}_j z) \prod_{j=1}^{\ell+k} (z + b_j)(\bar{b}_j z + 1),$$

where $b_{\ell+1} = \cdots = b_{\ell+k} = 0$.

A root-finding algorithm suitably designed for H-palindromic polynomials aims to compute such a structured factorization. The following result is at the basis of the derivation of our method. It generically describes the properties of a certain Diophantine equation associated with a given H-palindromic polynomial.

Theorem 1. *Let $s(z) \in \mathbb{C}[z]$ and $q(z) = a + bz + \bar{a}z^2 \in \mathbb{C}[z]$, $a \in \mathbb{C} \setminus \mathbb{R}$, be two nonzero H-palindromic polynomials with $n_s = 2(m-1)$ and $n_q = 2$, respectively, such that $s(z)$ and $q(z)$ are relatively prime. For any H-palindromic polynomial $p(z)$ with $n_p = 2m$ there exist uniquely determined a H-palindromic polynomial $t(z) \in \mathbb{C}[z]$ with $n_t = 2(m-1)$ and a T-palindromic polynomial $r(z) = r_0 + r_1 z + r_0 z^2 \in \mathbb{R}[z]$ such that*

$$p(z) = t(z)q(z) + r(z)s(z). \tag{2}$$