Journal of Computational and Applied Mathematics 272 (2014) 304–312

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Structured maximal perturbations for Hamiltonian eigenvalue problems

Paolo Buttà, Silvia Noschese *

Dipartimento di Matematica, SAPIENZA Università di Roma, P.le Aldo Moro 2, 00185 Rome, Italy

ARTICLE INFO

Article history:
Received 7 November 2012
Received in revised form 13 March 2013

Keywords:
Structured eigenvalue problem
Structured conditioning
Worst-case effect perturbation
Hamiltonian structure

ABSTRACT

We discuss the effect of structure-preserving perturbations on complex or real Hamiltonian eigenproblems and characterize the structured worst-case effect perturbations. We derive explicit expressions for the maximal Hamiltonian perturbations.

1. Introduction

For structured eigenvalue problems, algorithms that preserve the underlying matrix structure may improve the accuracy and efficiency of the eigenvalue computation and preserve possible eigenvalue symmetries in finite-precision arithmetic. The concept of backward stability, i.e., the requirement that the computed eigenvalues are the exact eigenvalues of a slightly perturbed matrix, is extended by requiring the perturbed matrix to have the same structure as the original one [1]. In order to assess the *strong* numerical backward-stability of the employed structured algorithm and not to overestimate the worst-case effect of perturbations, it is appropriate to consider suitable measures of the sensitivity of the eigenvalues to perturbations of the same structure.

Our analysis is based on the perturbation expansion of a simple eigenvalue. The structured condition number of an eigenvalue λ of a given matrix A is indeed a first-order measure of the worst-case effect on λ of perturbations of the same structure as A. Explicit expressions for structured norm-wise condition numbers with respect to the spectral norm can be found, e.g., in [2–4], whereas structured component-wise conditioning is analyzed, e.g., in [5,3]. The structured conditioning measures we deal with can be computed endowing the subspace of matrices with the Frobenius norm; see, e.g., [6–8,4] and references therein.

Here we are concerned with the Hamiltonian eigenvalue problems. They arise from a number of applications, particularly in systems and control theory. Algorithms and applications of Hamiltonian eigenproblems are discussed, e.g., in [9] and references therein.

We investigate the sensitivity of the eigenvalues of a complex [real] Hamiltonian matrix with respect to complex [real] Hamiltonian perturbations. Explicit expressions for the structured condition numbers can be found in [8]. They can be also used for the structured backward error analysis of structured algorithms; see, e.g., [10,11]. In this paper, attention has been paid to derive straightforwardly computable formulas for the maximal structured perturbations. Notice that, with the exception of purely imaginary eigenvalues, the (unstructured) maximal perturbation matrices do not have a Hamiltonian structure. A further motivation to derive structured worst-case effect perturbations is that they can be employed in the construction of dynamics for computing extremal points of structured ε-pseudospectra, see [12].

* Corresponding author. Tel.: +39 0649913200; fax: +39 0644701007.
E-mail addresses: butta@mat.uniroma1.it (P. Buttà), noschese@mat.uniroma1.it (S. Noschese).

0377-0427/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.cam.2013.04.031

It is well known, see, e.g., [13], that the worst (unstructured) perturbation which may affect a simple eigenvalue λ of a given matrix A arises under the action of the matrix yx^*, with x and y corresponding right and left eigenvectors, normalized to have $\|x\|_2 = \|y\|_2 = 1$ (hereafter, for any $m \times n$ matrix C, $C^* = \overline{C}^T$, where $\overline{C}$ [resp. C^T] denotes the conjugate [resp. transpose] of C). As a matter of fact, if we add the Hamiltonian-structure requirement, just as for sparsity-structures [14] or symmetry-patterns [15], we prove that the Hamiltonian matrix which yields the structured worst-case effect perturbation is the *structured analogue* of yx^*, i.e. the normalized projection of yx^* onto the space of the complex [real] Hamiltonian matrices. Notice that a further condition on the angle between the right and left eigenvectors is required here in the normalization.

The paper is organized as follows. Section 2 provides formulas for the maximal Hamiltonian perturbations. In Section 3, the real case is addressed. Numerical tests are presented in Section 4. Finally, conclusions are drawn in Section 5.

2. Eigenvalue structured conditioning of Hamiltonian matrices

We start with some notation and definitions. We denote by $\mathcal{H}_{\mathbb{C}}$ the subset of $2n$-dimensional complex Hamiltonian matrices, i.e.,

$$\mathcal{H}_{\mathbb{C}} = \left\{M \in \mathbb{C}^{2n\times 2n}: MJ = (MJ)^*\right\} = \left\{M = \begin{pmatrix} K & R \\ L & -K^* \end{pmatrix}: K, L, R \in \mathbb{C}^{n\times n},\ L = L^*,\ R = R^*\right\},$$

where J is the fundamental symplectic matrix, i.e.,

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

with I_n the $n \times n$ identity matrix. We note that the Hamiltonian matrix M is uniquely determined by the $4n^2$ real parameters given by the $2n^2$ real and imaginary parts of the elements of K, the $2n$ (real) diagonal elements of L, R, and the $2n(n-1)$ real and imaginary parts of the elements of the strict lower triangular parts of L, R.

Proposition 2.1. *The closest Hamiltonian matrix to a given matrix* $A = \begin{pmatrix} A_1 & A_3 \\ A_2 & A_4 \end{pmatrix} \in \mathbb{C}^{2n\times 2n}$ *with respect to the Frobenius norm is*

$$A|_{\mathcal{H}_{\mathbb{C}}} = \frac{1}{2}\begin{pmatrix} A_1 - A_4^* & A_3 + A_3^* \\ A_2 + A_2^* & A_4 - A_1^* \end{pmatrix} = \frac{1}{2}(A + JA^*J).$$

Proof. We observe that $M \to JM$ is an one to one map of $\mathcal{H}_{\mathbb{C}}$ on the space of Hermitian matrices which preserves the Frobenius norm, i.e., $\|JM\|_F = \|M\|_F$. By a well known result on the nearest Hermitian matrix [16], the nearest Hamiltonian matrix is given by

$$J\left(\frac{1}{2}((J^*A) + (J^*A)^*)\right) = \frac{1}{2}(A + JA^*J).$$

The proposition is thus proved. □

To state the next results we need some additional notation and definitions. It will be useful the following characterization of the space of complex Hamiltonian matrices,

$$\mathcal{H}_{\mathbb{C}} = \{H + \mathrm{i}\,W : H \in \mathcal{H}, W \in \mathcal{W}\},$$

where $\mathcal{H}$ [resp. $\mathcal{W}$] denotes the linear space of real Hamiltonian [resp. real skew-Hamiltonian] matrices, i.e.,

$$\mathcal{H} = \left\{H \in \mathbb{R}^{2n\times 2n}: HJ = (HJ)^T\right\},$$
$$\mathcal{W} = \left\{W \in \mathbb{R}^{2n\times 2n}: WJ = -(WJ)^T\right\}.$$

In particular, if $A = \Re(A) + \mathrm{i}\,\Im(A)$ then $A|_{\mathcal{H}_{\mathbb{C}}} = \Re(A)|_{\mathcal{H}} + \mathrm{i}\,\Im(A)|_{\mathcal{W}}$ where, for $B \in \mathbb{R}^{2n\times 2n}$,

$$B|_{\mathcal{H}} := \frac{1}{2}(B + JB^TJ) \qquad B|_{\mathcal{W}} := \frac{1}{2}(B - JB^TJ).$$

We finally introduce the normalized projection of a matrix A onto $\mathcal{H}_{\mathbb{C}}$ as

Theorem 2.2. *Let λ be a simple eigenvalue of a Hamiltonian matrix M, with corresponding right and left eigenvectors x and y normalized to have*

$$\|x\|_2 = \|y\|_2 = 1, \qquad \Im(y^*Jx) = 0. \tag{2.1}$$

Given any Hamiltonian matrix E with $\|E\|_F = 1$, let $\lambda_E(t)$ be an eigenvalue of $M + tE$ converging to λ as $t \to 0$. Then,

$$|\dot{\lambda}_E(0)| \le \max\left\{\left|\frac{y^*Gx}{y^*x}\right| : \|G\|_F = 1,\ G \in \mathcal{H}_{\mathbb{C}}\right\} = \frac{\|yx^*|_{\mathcal{H}_{\mathbb{C}}}\|_F}{|y^*x|}. \tag{2.2}$$