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Inverse eigenvalue problems for extended Hessenberg and extended tridiagonal matrices☆

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ABSTRACT

In inverse eigenvalue problems one tries to reconstruct a matrix, satisfying some constraints, given some spectral information. Here, two inverse eigenvalue problems are solved.

First, given the eigenvalues and the first components of the associated eigenvectors (called the weight vector) an extended Hessenberg matrix with prescribed poles is computed possessing these eigenvalues and satisfying the eigenvector constraints. The extended Hessenberg matrix is retrieved by executing particularly designed unitary similarity transformations on the diagonal matrix containing the eigenvalues. This inverse problem closely links to orthogonal rational functions: the extended Hessenberg matrix contains the recurrence coefficients given the nodes (eigenvalues), poles (poles of the extended Hessenberg matrix), and a weight vector (first eigenvector components) determining the discrete inner product. Moreover, it is also sort of the inverse of the (rational) Arnoldi algorithm: instead of using the (rational) Arnoldi method to compute a Krylov basis to approximate the spectrum, we will reconstruct the orthogonal Krylov basis given the spectral info.

In the second inverse eigenvalue problem, we do the same, but refrain from unitarity. As a result we execute possibly non-unitary similarity transformations on the diagonal matrix of eigenvalues to retrieve a (non)-symmetric *extended tridiagonal* matrix. The algorithm will be less stable, but it will be faster, as the extended tridiagonal matrix admits a low cost factorization of $\mathcal{O}(n)$ (n equals the number of eigenvalues), whereas the extended Hessenberg matrix does not. Again there is a close link with orthogonal function theory, the extended tridiagonal matrix captures the recurrence coefficients of bi-orthogonal rational functions. Moreover, it is again sort of inverse of the nonsymmetric Lanczos algorithm: given spectral properties, we reconstruct the two basis Krylov matrices linked to the nonsymmetric Lanczos algorithm.

1. Introduction

In this manuscript special instances of the following very general inverse eigenvalue problem are solved.

Definition 1.1 (*Inverse Eigenvalue Problem, IEP-general*)**.** Given n complex numbers λ_i and corresponding positive real weights w_i, $i = 1, 2, \ldots, n$. Without loss of generality, we will assume that the vector $w = [w_1, w_2, \ldots, w_n]$ has 2-norm

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equal to 1. Find a matrix M having a certain desired structure such that the eigenvalues of M are λ_i and such that the first component of the corresponding unit eigenvector is w_i.

This inverse eigenvalue problem computes the recurrence coefficients of orthogonal functions, orthogonal with respect to a discrete inner product with the λ_i as nodes and the $|w_i|^2$ as weights of the inner product. Solving such inverse eigenvalue problems, i.e. computing the recurrences and orthogonal functions (stemming from polynomials, Laurent polynomials, rational functions with finite and/or infinite poles, ...), given nodes and weights of the inner product, is typically done by plain matrix operations. By similarity transformations, one transforms the diagonal matrix of eigenvalues (see [1,2]) to a matrix of certain structure. The eigenvalues and eigenvectors link to the nodes and weights of to the inner product, and the matrix structure connects to the function type and eigenvalue distribution (e.g., Hessenberg vs. plain polynomials, Hermitian tridiagonal vs. polynomials with real eigenvalues, unitary Hessenberg vs. Szegő polynomials, extended Hessenberg without poles vs. Laurent polynomials, extended Hessenberg with poles vs. rational functions).

For a survey of methods on inverse eigenvalue problems, we refer to Chu and Golub [1], Boley and Golub [3], and see also the book of Golub and Meurant [4]. When the structure of the matrix M we are looking for is upper Hessenberg, taking the λ_i all on the real line, leads to the symmetry of this Hessenberg matrix. Hence, it becomes tridiagonal and is nothing else than the Jacobi matrix for the corresponding inner product, i.e., it gives the recurrence coefficients of the corresponding orthogonal polynomials [5]. The discrete least squares interpretations of these methods are presented by Reichel [6] and by Elhay, Golub, and Kautsky [7]. These methods efficiently exploit the tridiagonal structure of the matrix representing the recurrence relations and construct the optimal polynomial fitting in a least squares sense, given the function values in these real points λ_i. Based on the inverse unitary QR algorithm for computing unitary Hessenberg matrices [8], Reichel, Ammar, and Gragg [9] solve the approximation problem when the given function values are taken in points λ_i on the unit circle. Their algorithm is based on computational aspects associated with the family of polynomials orthogonal with respect to an inner product on the unit circle. Such polynomials are known as Szegő polynomials. Faßbender [10] presents an approximation algorithm based on an inverse unitary Hessenberg eigenvalue problem and shows that it is equivalent to computing Szegő polynomials. More properties of the inverse unitary Hessenberg eigenvalue problem are studied by Ammar and He [11].

A generalization of these ideas to vector orthogonal polynomials and to the least squares problems of a more general nature is presented by Bultheel and Van Barel in [12–14]. They developed an updating procedure to compute a sequence of orthonormal polynomial vectors with respect to that inner product where the points λ_i could lie anywhere in the complex plane. Again, if the inner products are prescribed in points on the real axis or on the unit circle, they present variants of the algorithm which are an order of magnitude more efficient. Similarly as in the scalar case, when all λ_i are real, the generalized Hessenberg becomes a banded matrix [15,16], and when all λ_i are on the unit circle, H can be parametrized using block Schur parameters [17]. Also a downdating procedure was developed [18]. For applications of downdating in data analysis, the reader can have a look at [19].

So far, we have only considered polynomial functions. When taking proper rational functions with prescribed poles $y_k \neq \infty$, $k = 1, \ldots, n$, the inverse eigenvalue problem becomes

$$Q^H D_z Q = S + D_y, \tag{1.1}$$

where D_y is the diagonal matrix based on the poles y_k (with an arbitrary value for y_0), and where S has to be lower semiseparable, i.e., all submatrices that can be taken out of the lower triangular part of S have rank at most 1. Also here, when all λ_i are real, S becomes a symmetric semiseparable matrix and when all λ_i lie on the unit circle, S has to be of lower as well as upper semiseparable form [20,21].

In this manuscript we will investigate general, not necessarily proper, rational functions. We will investigate the structure of the matrix that represents the recurrence coefficients for these sequences of orthogonal rational functions.

The techniques described above can be used in several applications in which polynomial or rational functions play an important role: linear system theory, control theory, system identification [22,23], data fitting [7], (trigonometric) polynomial least squares approximation [6,9], and so on. For a comprehensive overview of orthogonal rational functions, the interested reader can consult [24].

The article is organized as follows. There are two main sections, each discussing an inverse eigenvalue problem. Section 3 discusses the inverse eigenvalue problem for extended Hessenberg matrices: given eigenvalues and a vector of weights, construct via unitary similarity transformations an extended Hessenberg matrix, whose eigenvalues are as defined, and whose orthogonal eigenvectors have as first components the elements of the weight vector. In Section 4 we tackle an inverse eigenvalue problem where given two weight vectors and eigenvalues an extended tridiagonal matrix is constructed, whose eigenvalue decomposition has prescribed eigenvalues and the eigenvectors (not necessarily unitary anymore, but of unit length) have their first components related to the weight vectors. Both these sections are organized alike. First the Krylov subspace, whose orthogonal basis we would like to retrieve is presented and the structure of the matrix of recurrences is deduced. The compact representation of the matrix of recurrences is next presented, and will be used extensively in the algorithm design which relies heavily on basic 2×2 matrix operations. The description of the algorithm itself is subdivided in smaller parts, clearly distinguishing between finite and infinite poles.

We rely on the following notational conventions. Matrices are written as capitals A; the matrix element positioned on the intersection of row i and column j is denoted as a_{ij}. Vectors are typeset in bold: $\mathbf{v}$; the standard basis vectors are the $\mathbf{e}_i$'s. With $\cdot^T$ the transpose is meant; $\cdot^H$ stands for the Hermitian conjugate. Standard `Matlab` notation is used to select submatrices