Journal of Computational and Applied Mathematics 272 (2014) 399–416

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Symbol approach in a signal-restoration problem involving block Toeplitz matrices

Vincenza Del Prete [a], Fabio Di Benedetto [a,*], Marco Donatelli [b], Stefano Serra-Capizzano [b]

[a] Dipartimento di Matematica, Via Dodecaneso 35, 16146 Genova, Italy

[b] Dipartimento di Scienza e alta Tecnologia, Via Valleggio 11, 22100 Como, Italy

ARTICLE INFO

Article history:
Received 29 December 2012
Received in revised form 16 April 2013

Keywords:
Missing samples
Block Toeplitz matrix
Matrix valued symbol

ABSTRACT

We consider a special type of signal restoration problem where some of the sampling data are not available. The formulation related to samples of the function and its derivative leads to a possibly large linear system associated to a nonsymmetric block Toeplitz matrix which can be equipped with a 2×2 matrix-valued symbol. The aim of the paper is to study the eigenvalues of the matrix. We first identify in detail the symbol and its analytical features. Then, by using recent results on the eigenvalue distribution of block Toeplitz matrix-sequences, we formally describe the cluster sets and the asymptotic spectral distribution of the matrix-sequences related to our problem. The localization areas, the extremal behavior, and the conditioning are only observed numerically, but their behavior is strongly related to the analytical properties of the symbol, even though a rigorous proof is still missing in the block case.

1. Introduction and description of the problem

We consider a special type of signal restoration problem, where a finite number of samples of a band limited signal is lost. A band limited signal is a function belonging to the space B_ω of the square-integrable functions, whose Fourier transform is supported in $[-\omega, \omega]$. Functions belonging to this space can be represented by the Shannon series, which is an expansion in terms of the orthonormal basis of translates of the *sinc* function. The coefficients of the expansion, called *sampling formula*, are the samples of the function at a uniform grid on $\mathbb{R}$, with "density ω/π" (*Nyquist density*). This theory has been extended by replacing the orthonormal basis with more general families, like Riesz bases or frames, formed by translates of a single or more functions and other sampling formulas have been obtained (see the one-channel (4) and the two-channel (15) formulas and Ref. [1]). Frames, unlike Riesz or orthonormal bases, are overcomplete and their redundancy allows the recovery of missing samples.

The problem of the recovery of missing samples has been first investigated by Ferreira in [2] where the author shows that, under suitable oversampling assumptions, any finite set of samples can be recovered from the others. The recovery technique uses an oversampling one-channel formula (see (4)) and consists of solving a structured linear system $(I-S)X = B$ where S is a positive definite matrix, whose eigenvalues lie in the interval $(0, 1)$. If the missing samples are equidistant, this matrix is a scalar symmetric Toeplitz matrix associated to a scalar symbol (a characteristic function); thus, by standard localization and by distribution results [3], the eigenvalues lie in the convex hull of the symbol range and are clustered at the range.

* Corresponding author. Tel.: +39 010 3536883.
E-mail addresses: delprete@dima.unige.it (V. Del Prete), dibenede@dima.unige.it (F. Di Benedetto), marco.donatelli@uninsubria.it (M. Donatelli), stefano.serrac@uninsubria.it (S. Serra-Capizzano).

0377-0427/$ – see front matter
http://dx.doi.org/10.1016/j.cam.2013.05.018

In a further step, Santos and Ferreira considered the case of the two-channel derivative oversampling formula [4], in which a band limited function is expanded in terms of the translates of two functions (*generators*) (see (15)) and the coefficients are the samples of the function and its derivative. The authors of [4] show that a finite number of missing samples either of the function or of the derivative can be recovered, solving in each case a nonsingular linear system. Their technique extends in a natural way to the case where the missing samples come both from the function *and* its derivative, leading to a system $(I - S)X = B$, where the matrix S (see (19)) is a 2×2 block matrix depending on the generators and on the position of the missing samples. The unknowns X are the missing samples of the function and of its derivative.

In [1] Brianzi and Del Prete, using Ferreira's technique, studied the stability of the matrix in the two-channel derivative case, especially when the missing samples are located in positions

$$\mathcal{U} = \{m\, i_1, m\, i_2, \dots, m\, i_n\},$$

where m is an integer (*interleaving factor*). The practical interest for studying these cases lies in the technique of interleaving the samples of a signal, prior to their transmission or archival; the advantage of this procedure is that the transmitted (or stored) information becomes less sensitive to the burst errors that typically affect contiguous set of samples [2]. The authors have obtained estimates of the minimum and maximum eigenvalues of the block submatrices of the matrix S, showing that in some cases the matrix reduces to a (block) lower triangular matrix. Moreover they performed several numerical experiments on the dependence of the eigenvalues on the parameters of the problem and on the reconstruction of the signal, also in the ill-conditioned case of contiguous missing samples.

In this paper we consider the case of missing samples of the function and of its derivative at equidistant points. The formulation of the related signal restoration problem leads to a nonsymmetric block Toeplitz matrix, possibly large. Here we perform three steps. First we construct an explicit expression of the symbol f, by interpreting the entries of the matrix as Fourier coefficients of the symbol. As a second step, we study the related symbol and, as a third step, we use the information for giving a spectral characterization of the matrices S in terms of the different parameters. More specifically, concerning the second step, we prove that the double channel matrix is similar (via a permutation transform) to a standard block Toeplitz matrix $T_n(f)$ with 2-by-2 nonsymmetric matrix-valued symbol $f(\theta)$ associated with two main parameters: the oversampling parameter $r \in (0, 1)$ (see (13)) and the interleaving factor m (positive integer), both depending on the problem. For any choice of the parameters the eigenvalues $\lambda_1(f(\theta))$ and $\lambda_2(f(\theta))$ of $f(\theta)$ have the following features:

(i) their range is real (proved formally),
(ii) their range consists of two intervals contained in $(0, 1)$ (proved formally only for special sets of the parameters).

For specific choices of the parameters it can also be seen that the range of $\lambda_1(f(\theta))$ is made by two single points V_1, V_2 and the range of $\lambda_2(f(\theta))$ is made by two single points V_3, V_4 with $0 < V_1 \le V_2 \le V_3 \le V_4 < 1$ and where the Lebesgue measure of the θ-values such that $\lambda(f(\theta)) = V_j$ is a positive number M_j for every $j = 1, 2, 3, 4$. In addition this behavior of the symbol (though not proved rigorously for every possible choice of the parameters) seems to be general. On the other hand, the numerical tests show that

1. all the eigenvalues of $T_n(f)$ are contained in (V_1, V_4);
2. the global spectrum of the sequence $\{T_n(f)\}$ is distributed as the symbol f (see Theorems 2.3 and 5.1) and it is clustered to the set $\{V_1, V_2, V_3, V_4\}$ and the number of the eigenvalues which are close within ϵ to V_j is equal to $2M_jn + o(n)$ (the quantity named $o(n)$ seems to grow only logarithmically with n) where the M_j's are the Lebesgue measures mentioned above with $M_1 + M_2 + M_3 + M_4 = 2\pi$ i.e. the total measure of the definition set of the symbol;
3. the minimal eigenvalue of $T_n(f)$ converges to V_1 monotonically and exponentially with respect to n;
4. the maximal eigenvalue of $T_n(f)$ converges to V_4 monotonically and exponentially with respect to n;
5. as the parameter m that characterizes the symbol f goes to infinity, the values V_1 and V_2 tend to r^2 and the other two V_3 and V_4 tend to $2r - r^2$.

Now a brief discussion on these items is required since a lot of information can be extracted from them. As an example, if one puts together the second and the fifth items we conclude that, for m large enough, the symbol behaves as a step function, exactly as in the case of a one-channel problem. Whereas the statements contained in the second and the fifth items are proven rigorously (the second in Theorem 5.1 by using Szegö-like results by Donatelli, Neytcheva, and Serra-Capizzano [5], the fifth in the derivations before Section 5.1), the first, the third, and the fourth are only observed in practical simulations. However the above three statements carry a strong information that could be used theoretically: in fact the observed behavior in items 1, 3, 4 is typical of a Hermitian-valued symbol [6] so that one of the following alternatives has to be true:

*s*1 (first alternative conjecture) We know that the symbol $f(\theta)$ is nonsymmetric, but it is symmetrizable i.e. there exists a matrix $Q(\theta)$ such that $Q(\theta)f(\theta)Q(\theta)^{-1}$ is symmetric. We would like to prove that the matrix $Q(\theta)$ could be chosen as a constant matrix independent of θ: however this seems unlikely.

*s*2 (second alternative conjecture) There exist cases in which the statement *s*1 is not true, but the properties 1, 3, 4 are verified. In other words, the assumption of a constant transform can be weakened to prove properties 1, 3, 4.