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On condition numbers of the spectral projections associated with periodic eigenproblems

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ABSTRACT

In this paper, we study and analyze absolute and relative condition numbers of the spectral projections for regular periodic eigenproblems. The main contribution is to derive explicit expressions of the condition numbers of the j-th left and right spectral projections. Numerical examples are given to illustrate the proposed condition numbers for the spectral projections associated with periodic eigenproblems.

1. Introduction

In this paper we study condition numbers for the spectral projections of the multivariate eigenproblem:

$$\begin{pmatrix} \alpha_1E_1 & 0 & \cdots & 0 & -\beta_1A_1 \\ -\beta_2A_2 & \alpha_2E_2 & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\beta_pA_p & \alpha_pE_p \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_p \end{pmatrix} \equiv C\begin{pmatrix} \alpha_1,\dots,\alpha_p \\ \beta_1,\dots,\beta_p \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} = 0, \tag{1.1}$$

where $A_j, E_j \in \mathbb{C}^{n\times n}$, β_j, α_j are complex variables and $x_j \neq 0 \in \mathbb{C}^n$ for $j = 1, \dots, p$. The periodic matrix pairs $\{(A_j, E_j)\}_{j=1}^p$ are called regular if

$$\det\left[C\begin{pmatrix} \alpha_1,\dots,\alpha_p \\ \beta_1,\dots,\beta_p \end{pmatrix}\right] \not\equiv 0.$$

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Let the matrix pairs $\{(A_j, E_j)\}_{j=1}^p$ be regular. If there are complex numbers $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_p$ with $(\prod_{j=1}^p \alpha_j, \prod_{j=1}^p \beta_j) \neq (0, 0)$ satisfying

$$\det\left[C\begin{pmatrix}\alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_p\end{pmatrix}\right] = 0,$$

then we say that $(\prod_{j=1}^p \alpha_j, \prod_{j=1}^p \beta_j)$ is an eigenvalue pair of $\{(A_j, E_j)\}_{j=1}^p$. If (π_α, π_β) is an eigenvalue pair of $\{(A_j, E_j)\}_{j=1}^p$, then (π_α, π_β) and $(\tau\pi_\alpha, \tau\pi_\beta)$ represent the same eigenvalue for any nonzero τ. If $\pi_\beta \neq 0$ then $\lambda = \pi_\alpha/\pi_\beta$ is a finite eigenvalue; otherwise $(\pi_\alpha, 0)$ represents an infinite eigenvalue. The set of all eigenvalue pairs of $\{(A_j, E_j)\}_{j=1}^p$ is denoted by $\lambda(\{(A_j, E_j)\}_{j=1}^p)$. The detailed discussion of the eigenvalue problem for regular periodic matrix pairs can be found in [1–4].

When $\{(A_j, E_j)\}_{j=1}^p$ is regular, there exist unitary matrices $U_j, V_j \in \mathbb{C}^{n\times n}$ such that [1,4]

$$A_j = U_j\begin{pmatrix}A_{11}^{(j)} & A_{12}^{(j)} \\ 0 & A_{22}^{(j)}\end{pmatrix}V_{j-1}^*, \qquad E_j = U_j\begin{pmatrix}E_{11}^{(j)} & E_{12}^{(j)} \\ 0 & E_{22}^{(j)}\end{pmatrix}V_j^*, \quad j = 1, \ldots, p, \tag{1.2}$$

where $V_0 = V_p$, $A_{11}^{(j)}, E_{11}^{(j)} \in \mathbb{C}^{m\times m} (m < n)$, and $\cdot^*$ denotes a conjugate transpose of a matrix. Assume $\lambda(\{(A_{11}^{(j)}, E_{11}^{(j)})\}_{j=1}^p) \cap \lambda(\{(A_{22}^{(j)}, E_{22}^{(j)})\}_{j=1}^p) = \emptyset$. Then the following periodic generalized coupled Sylvester equation

$$\begin{cases} A_{11}^{(j)}X_{j-1} - Y_jA_{22}^{(j)} = -A_{12}^{(j)}, \\ E_{11}^{(j)}X_j - Y_jE_{22}^{(j)} = -E_{12}^{(j)}, \end{cases} \quad j = 1, \ldots, p \tag{1.3}$$

has a unique solution $\{(X_j, Y_j)\}_{j=1}^p$, where $X_0 = X_p$, see [5]. By setting

$$\begin{aligned} S_j &= U_j\begin{pmatrix}I_m & Y_j \\ 0 & I_{n-m}\end{pmatrix} = (S_1^{(j)}, S_2^{(j)}), \qquad T_j = S_j^{-1} = \begin{pmatrix}I_m & -Y_j \\ 0 & I_{n-m}\end{pmatrix}U_j^* = \begin{pmatrix}T_1^{(j)} \\ T_2^{(j)}\end{pmatrix}, \\ G_j &= V_j\begin{pmatrix}I_m & X_j \\ 0 & I_{n-m}\end{pmatrix} = (G_1^{(j)}, G_2^{(j)}), \qquad H_j = G_j^{-1} = \begin{pmatrix}I_m & -X_j \\ 0 & I_{n-m}\end{pmatrix}V_j^* = \begin{pmatrix}H_1^{(j)} \\ H_2^{(j)}\end{pmatrix}, \end{aligned} \tag{1.4}$$

where $S_1^{(j)}, G_1^{(j)} \in \mathbb{C}^{n\times m}$ and $T_1^{(j)}, H_1^{(j)} \in \mathbb{C}^{m\times n}$, we obtain

$$A_j = S_j\begin{pmatrix}A_{11}^{(j)} & 0 \\ 0 & A_{22}^{(j)}\end{pmatrix}G_{j-1}^{-1}, \qquad E_j = S_j\begin{pmatrix}E_{11}^{(j)} & 0 \\ 0 & E_{22}^{(j)}\end{pmatrix}G_j^{-1}, \quad j = 1, \ldots, p. \tag{1.5}$$

This means that $\mathscr{R}(S_1^{(j)})$ and $\mathscr{R}(G_1^{(j)})$ are the j-th simple left and right periodic deflating subspaces of $\{(A_j, E_j)\}_{j=1}^p$ corresponding to $\lambda(\{(A_{11}^{(j)}, E_{11}^{(j)})\}_{j=1}^p)$, see [4]. For $j = 1, \ldots, p$, the $n \times n$ matrices

$$P_l^{(j)} = S_j\begin{pmatrix}I_m & 0 \\ 0 & 0\end{pmatrix}S_j^{-1}, \qquad P_r^{(j)} = G_j\begin{pmatrix}I_m & 0 \\ 0 & 0\end{pmatrix}G_j^{-1} \tag{1.6}$$

are respectively the j-th left and right spectral projections onto the j-th simple right and left deflating subspaces of the periodic matrix pairs $\{(A_j, E_j)\}_{j=1}^p$ corresponding to $\lambda(\{(A_{11}^{(j)}, E_{11}^{(j)})\}_{j=1}^p)$.

In the literature, there are many papers studying the perturbation theory and numerical methods of spectral projections (see for example, [3,6–9]). The spectral projections of regular periodic matrix pairs play an important role in computing periodic reachability and observability Gramians [3]. For example, a periodic descriptor system with time-varying dimensions:

$$E_jx_{j+1} = A_jx_j + B_ju_j, \qquad y_j = C_jx_j, \quad j = 1, \ldots, p,$$

where the rectangular matrices A_j, E_j, B_j, and C_j are periodic with a period $p \geq 1$, see [3]. Assume that regular periodic matrix pairs $\{(A_j, E_j)\}_{j=1}^p$ are stable, that is, all their finite eigenvalues of $\{(A_j, E_j)\}_{j=1}^p$ lie inside the unit circle, then the causal reachability Gramians $\{G_j^{cr}\}_{j=1}^p$ refer to the unique Hermitian positive semidefinite solution of the following projected generalized discrete-time periodic Lyapunov equations:

$$E_jG_{j+1}^{cr}E_j^T - A_jG_j^{cr}A_j^T = P_l^{(j)}B_jB_j^*P_l^{(j)*}, \qquad G_j^{cr} = P_r^{(j)}G_j^{cr}P_r^{(j)*}, \quad j = 1, \ldots, p,$$

where $P_l^{(j)}$ and $P_r^{(j)}$ are the j-th left and right spectral projections corresponding to the finite eigenvalues of $\{(A_j, E_j)\}_{j=1}^p$. For a matrix and a regular matrix pair, Sun [8,9] derived explicit expressions of condition numbers for spectral projections. To the best of our knowledge, similar results have not been developed for periodic eigenproblems. The purpose of this paper is to define the absolute and relative condition numbers of spectral projections onto the j-th left and right deflating subspaces of the regular periodic matrix pairs appropriately, and to derive explicit expressions of these condition numbers.