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Existence of oscillatory solutions of second order delay differential equations

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ABSTRACT

In this paper, we investigate existence of oscillatory solutions for a forced second order nonlinear delay differential equations

$$[r(t)\Phi(x'(t))]' + \sum_{i=1}^{m} f_i(t, x(g_i(t))) = q(t),$$

where $f_i \in C([t_0, \infty)\times R, R)$, $g_i(t) \le t$, $\lim_{t\to\infty} g_i(t) = \infty$, $i = 1, 2, \cdots, m$, $\Phi \in C^1(R, R)$, $\Phi(u)$ is an increasing function for all $u \in R$, $\Phi^{-1}(u)$ satisfies the local *Lipischitz* condition. A new sufficient condition for global existence of oscillatory solution is obtained by the Schauder–Tychonoff theorem. When $\Phi(u) = u^\alpha$ with $\alpha \ge 1$ being the ratio of two positive odd integers has also been studied. We give examples to illustrate the applicability of our results.

1. Introduction

In this paper we study the existence of oscillatory solutions for the nonlinear second order delay differential equations with forced term

$$[r(t)\Phi(x'(t))]' + \sum_{i=1}^{m} f_i(t, x(g_i(t))) = q(t), \quad t \ge t_0. \tag{1}$$

Under the following conditions:

(1) $r \in C^1\big([t_0, \infty), R^+\big)$, $q, g_i \in C\big([t_0, \infty), R\big)$, $f_i \in C\big([t_0, \infty) \times R, R\big)$, and $g_i(t) \le t$, $\lim_{t\to\infty} g_i(t) = \infty$, $i = 1, 2, \ldots, m$;
(2) $\Phi \in C^1\big(R, R\big)$, $\Phi(u)$ is increasing function for all $u \in R$, $\Phi^{-1}(u)$ satisfies the local *Lipischitz* condition.

During the past three decades, the investigation of oscillatory theory for delay differential equations and delay dynamic equations has attracted attention of numerous researchers due to their significance in theory and applications. We mention here the monographs of A.D. Myshkis [1], N.V. Shevelo [2] and R.P. Agarwal, L. Berezansky and E. Braverman [3]. The oscillation properties of second order delay differential equations were considered also in R.G. Koplatadze, G. Kvinikadze

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and I.P. Stavroulakis [4], M.G. Shmul'yan [5]. Distances between adjacent zeros of oscillating solutions are estimated in [6,7] for delay and for neutral second order equation in [8]. Distances between zero of solution and zero of its derivative were estimated in [9]. Based on oscillation properties, asymptotic properties of second order delay differential equations were studied in [10]. For related work, we refer the reader to Refs. [11–22]. However, to the best of our knowledge, the existence of oscillatory solutions for differential equation has been scarcely investigated. Thus, the research presents its significance.

As usual, a solution of Eq. (1) is a function $x(t)$ defined on $[r_\sigma, \infty)$ such that $x(t)$ and $r(t)\Phi(x'(t))$ are continuously differentiable on $[r_\sigma, \infty)$ and $[\sigma, \infty)$. Our attention will be restricted to those solutions $x(t)$ of (1) which satisfy $\sup|x(t)| > 0$, for $t \geq T \geq \sigma$. Such a solution is said to be oscillatory if it has a sequence of zeros tending to infinity. Otherwise, it is said to be nonoscillatory.

The purpose of this paper is to prove a general result for Eq. (1) on the existence of oscillatory solutions. Here, for any $\sigma \geq t_0$, let $r_\sigma = \min_{1\leq i\leq m} \inf_{t\geq\sigma} g_i(t)$.

Throughout this paper, we will use the following notations. For a constant $\gamma > 0$,

$$p_i(t)_\gamma = \max_{|x|\leq\gamma} \frac{1}{\gamma} |f_i(t,x)|, \quad t \geq t_0,\ i = 1, 2, \ldots, m.$$

L_γ denotes the local lipischitz constants of functions $\Phi^{-1}(u)$.

2. The main results

Lemma (*See [11]*)**.** *Let X be a locally convex space, $K \subset X$ be nonempty and convex, $S \subset K$, S be compact. Given a continuous map $F : K \to S$, then there exists $\tilde{x} \in S$ such that $F(\tilde{x}) = \tilde{x}$.*

Theorem. *Assume that there exist $\eta, \gamma > 0$ such that $r(t) > \eta$,*

$$\frac{1}{r(t)} \int_t^\infty q(s)ds \text{ is integrable on } [t_0, \infty), \tag{2}$$

$$\frac{1}{r(t)} \int_t^\infty \sum_{i=1}^m p_i(s)_\gamma ds \text{ is integrable on } [t_0, \infty), \tag{3}$$

moreover, there exist two sequences $\{t_n\}$, $\{s_n\}$ with $t_n \to \infty$, $s_n \to \infty$, and such that

$$\int_{t_n}^\infty \Phi^{-1}\left(\frac{1}{r(s)} \int_s^\infty \left(q(\tau) + \gamma \sum_{i=1}^m p_i(\tau)_\gamma\right) d\tau\right) ds < 0, \tag{4}$$

and

$$\int_{s_n}^\infty \Phi^{-1}\left(\frac{1}{r(s)} \int_s^\infty \left(q(\tau) - \gamma \sum_{i=1}^m p_i(\tau)_\gamma\right) d\tau\right) ds > 0. \tag{5}$$

Then (1) has an oscillatory solution $x(t)$ defined on $[t_0, \infty)$ with $|x| \leq \gamma$, and $\lim_{t\to\infty} x(t) = 0$.

Proof. The proof is based on an application of the well known Schauder–Tychonoff fixed point theorem. From (2) and (3), for any $\gamma > 0$, we can choose a large $T_\gamma \geq T$ such that for all $t \geq T_\gamma$,

$$\int_t^\infty \Phi^{-1}\left(\frac{1}{r(s)} \int_s^\infty \left(q(\tau) + \gamma \sum_{i=1}^m p_i(\tau)_\gamma\right) d\tau\right) ds \leq \gamma, \tag{6}$$

and

$$\int_t^\infty \Phi^{-1}\left(\frac{1}{r(s)} \int_s^\infty \left(q(\tau) - \gamma \sum_{i=1}^m p_i(\tau)_\gamma\right) d\tau\right) ds \geq -\gamma. \tag{7}$$

Let $C[T_0, \infty)$ denote the locally convex space of all continuous functions with topology of uniform convergence on compact subsets of $[T_0, \infty)$, where $T_0 = \min_{1\leq i\leq m} \inf_{t\geq T} g_i(t)$.

Let $S = \{x \in C[T_0, \infty), |x(t)| \leq \gamma\}$. Clearly, S is a closed convex subset of $C[T_0, \infty)$.

Introduce an operator F by,

$$(Fx)(t) = \begin{cases} \int_t^\infty \Phi^{-1}\left(\frac{1}{r(s)} \int_s^\infty \left(q(\tau) - \sum_{i=1}^m f_i(\tau, x(g_i(\tau)))\right) d\tau\right) ds & t > T_\gamma, \\ (Fx)(T_\gamma), \quad T_0 \leq t \leq T_\gamma. \end{cases}$$

It is easy to see that for any $x \in S$, $(Fx)(t)$ is well defined on $[T_0, \infty)$ continuously.