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A sharp error bound of the approximate solutions for saddle point linear systems

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ABSTRACT

Sharp error estimations for numerical solutions of saddle point linear systems are presented. Based on these estimations, fast algorithms for computing error bounds of the numerical solutions are proposed. The error bounds obtained by these algorithms are "verified" in the sense that all the possible rounding errors have been taken into account. Techniques for accelerating the computation and obtaining smaller error bounds are introduced. Numerical results show the properties of the proposed algorithms.

1. Introduction

In this paper, we are concerned with the accuracy of numerically computed solutions of the saddle point linear systems

$$\mathcal{H}u = b, \quad \mathcal{H} := \begin{pmatrix} A & B^T \\ B & -C \end{pmatrix}, \; u := \begin{pmatrix} x \\ y \end{pmatrix}, \; b := \begin{pmatrix} f \\ g \end{pmatrix}, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{m\times m}$, $f \in \mathbb{R}^n$ and $g \in \mathbb{R}^m$ are given, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ are to be solved, $n \geq m$, A is symmetric positive definite (SPD), B has full rank, and C is symmetric positive semi-definite (SPSD), which imply $\mathcal{H}$ is nonsingular. The systems (1) appear in the various science and engineering applications, including partial differential equations and optimization problems.

Let $u^* = (x^{*^T}, y^{*^T})^T$ and $\tilde{u} = (\tilde{x}^T, \tilde{y}^T)^T$ denote the exact and numerical solutions of (1), respectively. We consider in this paper numerical verification of $\tilde{u}$, specifically, computing rigorous upper bounds of $\|\tilde{u} - u^*\|_\infty$ using floating point operations.

Pioneering work is the result by Chen and Hashimoto [1]. They skillfully exploited the special structure of (1). This result enables to avoid computing an approximation of $\mathcal{H}^{-1}$. Let

$$\begin{pmatrix} r_f \\ r_g \end{pmatrix} := \begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} - \begin{pmatrix} f \\ g \end{pmatrix}$$

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be residual vectors. They presented the error estimations

$$\|\tilde{x}-x^*\|_2 \le \|A^{-1}\|_2(\|r_f\|_2 + \|B^T\|_2\|\tilde{y}-y^*\|_2), \tag{2}$$

$$\|\tilde{y}-y^*\|_2 \le \zeta(\|BA^{-1}\|_2\|r_f\|_2 + \|r_g\|_2), \tag{3}$$

where

$$\zeta := \frac{\|A\|_2\|(BB^T)^{-1}\|_2}{1+\|A\|_2\|(BB^T)^{-1}\|_2\sigma_{\min}(C)}$$

and $\sigma_{\min}(C)$ denotes the smallest singular value of C. Substituting (3) into (2), we obtain

$$\|\tilde{x}-x^*\|_2 \le \|A^{-1}\|_2((1+\zeta\|B^T\|_2\|BA^{-1}\|_2)\|r_f\|_2 + \zeta\|B^T\|_2\|r_g\|_2). \tag{4}$$

We can obtain the upper bounds of $\|\tilde{u}-u^*\|_\infty$ based on (3) and (4), since $\|\tilde{u}-u^*\|_\infty = \max(\|\tilde{x}-x^*\|_\infty, \|\tilde{y}-y^*\|_\infty) \le \max(\|\tilde{x}-x^*\|_2, \|\tilde{y}-y^*\|_2)$ holds. An important special case is for $C=0$. In this case, we have $\sigma_{\min}(C)=0$, so that (3) and (4) give

$$\|\tilde{x}-x^*\|_2 \le \|A^{-1}\|_2((1+\|A\|_2\|B^T\|_2\|(BB^T)^{-1}\|_2\|BA^{-1}\|_2)\|r_f\|_2 + \|A\|_2\|B^T\|_2\|(BB^T)^{-1}\|_2\|r_g\|_2), \tag{5}$$

$$\|\tilde{y}-y^*\|_2 \le \|A\|_2\|(BB^T)^{-1}\|_2(\|BA^{-1}\|_2\|r_f\|_2 + \|r_g\|_2). \tag{6}$$

They proposed verification algorithms based on (5) and (6).

Hashimoto [2] also treated the case of $C=0$, and improved (5) and (6). He presented the estimations

$$\|\tilde{x}-x^*\|_2 \le \|A^{-1}\|_2\|r_f\|_2 + \kappa(A)\|L_B^{-1}r_g\|_2, \tag{7}$$

$$\|L_B^T(\tilde{y}-y^*)\|_2 \le \kappa(A)\|r_f\|_2 + \|A\|_2\|L_B^{-1}r_g\|_2, \tag{8}$$

where L_B is an $m\times m$ nonsingular matrix satisfying $L_BL_B^T = BB^T$ and $\kappa(A) := \|A\|_2\|A^{-1}\|_2$. Let $L_B^{-T} := (L_B^{-1})^T$. From (8), we have

$$\|\tilde{y}-y^*\|_2 \le \|L_B^{-T}\|_2\|L_B^T(\tilde{y}-y^*)\|_2 \le \|L_B^{-1}\|_2(\kappa(A)\|r_f\|_2 + \|A\|_2\|L_B^{-1}r_g\|_2). \tag{9}$$

Kimura and Chen [3] treated the case of $C=0$ and A being SPSD. They effectively utilized a preconditioner and proposed verification algorithms. This result also enables us to avoid computing the approximation of $\mathcal{H}^{-1}$. Their algorithms are applicable also in the case of $C=0$ and A being SPD.

The purpose of this paper is to present and propose error estimations and verification algorithms for the solution of (1), respectively. Since A is SPD, there exists a nonsingular $n\times n$ matrix L_A satisfying $L_AL_A^T = A$. Let $L_A^{-1}B^T = QR$ be the thin QR factorization of $L_A^{-1}B^T$, where $Q\in\mathbb{R}^{n\times m}$ is column-orthogonal and $R\in\mathbb{R}^{m\times m}$ is upper triangular. Since L_A is nonsingular and B has full rank, R is also nonsingular. We derive the estimations

$$\|\tilde{x}-x^*\|_2 \le \|L_A^{-1}\|_2\|L_A^{-1}r_f\|_2 + \eta\|A^{-1}B^T\|_2\|r_g\|_2, \tag{10}$$

$$\|\tilde{y}-y^*\|_2 \le \eta(\|BA^{-1}r_f\|_2 + \|r_g\|_2), \tag{11}$$

where

$$\eta := \frac{\|R^{-1}\|_2^2}{1+\|R^{-1}\|_2^2\sigma_{\min}(C)}.$$

Let $R^{-T} := (R^{-1})^T$. For $C=0$, we present

$$\|\tilde{x}-x^*\|_2 \le \|L_A^{-1}\|_2(\|L_A^{-1}r_f\|_2 + \|R^{-T}r_g\|_2), \tag{12}$$

$$\|\tilde{y}-y^*\|_2 \le \|R^{-T}\|_2\|L_A^{-1}r_f\|_2 + \|R^{-1}R^{-T}r_g\|_2. \tag{13}$$

Let ε_{CH}, δ_{CH}, δ_{CH_0}, ε_{CH_0}, δ_H, ε_H, δ_M, ε_M, δ_{M_0} and ε_{M_0} be the right hand sides of (3), (4), (5), (6), (7), (9), (10), (11), (12) and (13), respectively. We prove $\delta_M \le \delta_{CH}$, $\varepsilon_M \le \varepsilon_{CH}$, $\delta_{M_0} \le \delta_H \le \delta_{CH_0}$ and $\varepsilon_{M_0} \le \varepsilon_H \le \varepsilon_{CH_0}$ to show the advantages of (10)–(13), and propose the verification algorithms based on (12) and (13). These algorithms do not assume but prove that A is SPD and B has full rank. We introduce techniques for accelerating the verification and obtaining smaller error bounds.

This paper is organized as follows: In Section 2, notation and theories used in this paper are introduced. In Section 3, (10), (11), (12), (13), $\delta_M \le \delta_{CH}$, $\varepsilon_M \le \varepsilon_{CH}$, $\delta_{M_0} \le \delta_H \le \delta_{CH_0}$ and $\varepsilon_{M_0} \le \varepsilon_H \le \varepsilon_{CH_0}$ are proved. In Section 4, a theory for computing upper bounds of $\|\tilde{x}-x^*\|_2$ and $\|\tilde{y}-y^*\|_2$ based on (12) and (13), respectively, is established. In Sections 5 and 6, the techniques for accelerating the verification and obtaining smaller error bounds are introduced, respectively. In Section 7, numerical results are reported. Section 8 finally summarizes the results in this paper and highlights possible extension and future work.