Journal of Computational and Applied Mathematics 277 (2015) 94–105

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On the choice of the frequency in trigonometrically-fitted methods for periodic problems

Jesús Vigo-Aguiar [a], Higinio Ramos [a,b,*]

[a] Scientific Computing Group, University of Salamanca, Spain

[b] Escuela Politécnica Superior de Zamora, University of Salamanca, Spain

ARTICLE INFO

Article history:
Received 16 July 2013
Received in revised form 22 July 2014

MSC:
65L05

Keywords:
Trigonometrically fitted methods
Nonlinear periodic oscillators
Frequency determination
First integrals
Total energy minimization

ABSTRACT

In this paper the use of a trigonometrically fitted method to obtain the approximate solutions of some nonlinear periodic oscillators is presented. A great number of different approaches have been considered to obtain analytical approximations for this kind of problems: a generalized decomposition method (GDM), a linearized harmonic balance procedure (LHB), the homotopy perturbation method (HPM), the harmonic balanced method (HBM), the Adomian decomposition method, etc. From those approaches, analytical approximations to the frequency of oscillation and periodic solutions are obtained, which are valid for a large range of amplitudes of oscillation. However, these techniques have been limited to obtain only one or two iterates because of the great amount of algebra involved. We use a trigonometrically adapted method to obtain numerical approximations to the solutions, yielding very acceptable results, on the basis that the approximation of the frequency of the method is done with great accuracy. There are a lot of trigonometrically fitted methods in the literature, but there is not a definite way to obtain the optimal value of the frequency. We present a strategy for the choice of the parameter value in the adapted method based on the minimization of the total energy. Some examples solved numerically confirm the good performance of the adopted strategy.

1. Introduction

Many nonlinear phenomena can be described in terms of nonlinear oscillators. Perhaps one of the most known nonlinear systems is the simple pendulum (a particle of mass m attached to a wall by a string moving on a surface without friction), which for small angle oscillations the periodic motion is harmonic; nevertheless, for large oscillations the period depends on the amplitude [1].

Nonlinear oscillators appear in quantum mechanics, biology, optics, and of course, in classical mechanics. Recently, a lot of work has been devoted to obtaining approximate analytical solutions to nonlinear oscillators, and their periods of vibration. A great number of different approaches have been considered to obtain analytical approximations for these oscillators: the generalized decomposition method (GDM), the linearized harmonic balance procedure (LHB), the homotopy perturbation method (HPM), the harmonic balanced method (HBM), the Adomian decomposition method, etc. Most of the above procedures consist in transforming the given second order initial-value problem in an infinite sequence of linear inhomogeneous second order initial-value problems. A survey of these approaches with a lot of references can be found

* Corresponding author at: Scientific Computing Group, University of Salamanca, Spain. Tel.: +34 980545000; fax: +34 980545002.
E-mail address: higra@usal.es (H. Ramos).

http://dx.doi.org/10.1016/j.cam.2014.09.008
0377-0427/

in [2]. But for most cases, the application of these methods leads to very complicated sets of algebraic equations with very complex nonlinearities. In this paper we consider the use of trigonometrically adapted methods for solving this kind of problems, but on the basis that an appropriate strategy is used to obtain the value of the involved parameter with great accuracy.

Consider a nonlinear oscillator whose trajectory $x(t)$ is a solution of Newton's equation of motion given in the form

$$\ddot{x} = -\nabla(V(x)) \tag{1}$$

where the potential-energy function $V(x)$ verifies that $f(x) = \frac{-dV(x)}{dx}$ for a conservative force $f(x)$. As is commonly used in classical mechanics the dot stands for the derivative with respect to time. In the above differential equation it could appear possible constants, that we set equal to one. This does not change in any way the essential features of the solutions [3]. The solutions of this equation may present different behaviors. In case of bounded motion, assuming that the equilibrium position takes place at $x = 0$, the amplitude is the largest distance from this equilibrium position. If $f(x)$ does not have a dominant term proportional to x, then the equation in (1) is named as "truly nonlinear oscillator" [3].

An important feature in case of a periodic solution of a nonlinear oscillator is the period, that is, the smallest real value $T > 0$ for which $x(t) = X(t + T)$.

The use of analytical, qualitative, and numerical techniques helps to determine if the solution of a particular problem is periodic or not. It is important to note that the frequency of the adapted method (the parameter ω that will appear later) is in general different from the angular frequency of the motion $\Omega = \frac{2\pi}{T}$ (see [4]). In this context the accurate determination of the value for ω is of high concern.

In a later section we provide a range of worked examples illustrating different behaviors. We consider two trigonometrically adapted methods of Falkner-type and use these methods with a suitable strategy for the parameter determination to obtain approximate solutions to the problems. The numerical results show the good performance of the proposed strategy.

2. Some background

In this context it is important to determine whether a given differential equation has periodic solutions or not. For this purpose it is appropriate to reformulate the differential equation in (1) as a system of two first order equations in the form

$$\dot{x} = y, \qquad \dot{y} = f(x) \tag{2}$$

where the variables x and y define the phase–space. The equilibrium solutions are of the form

$$x(t) = \bar{x}, \qquad y(t) = 0$$

where $\bar{x}$ are constant solutions of $f(x) = 0$. If we consider initial values $x(0) = x_0, y(0) = y_0$, for $t \geq 0$ the points $(x(t), y(t))$, where $x(t)$ and $y(t)$ are the corresponding solutions of the IVP with equations given in (2), describe a trajectory in the phase–space. The closed curves in the phase–space plane correspond to periodic solutions. The differential equation of the trajectories is readily deduced from (2) and is given by [3]

$$\frac{dy}{dx} = \frac{f(x)}{y}.$$

Taking initial conditions $x(0) = A, y(0) = 0$, after integrating we obtain the first integral

$$\frac{1}{2}y^2 + V(x) = V(A) = K$$

where the constant total energy function is given by $H(x, y) = \frac{1}{2}y^2 + V(x)$.

In case of a periodic motion, considering the values on the trajectory with $y = 0$, which result to be the two values x_+ and x_- for which it is $V(x_+) = V(x_-) = K$, the period of the motion is given by [5]

$$T = \sqrt{2}\int_{x_-}^{x_+} \frac{dx}{\sqrt{K - V(x)}} \tag{3}$$

and the angular frequency is $\Omega = \frac{2\pi}{T}$. In a few cases it is possible to solve this integral analytically and to obtain the exact period. But in most cases only an approximate solution may be found. Several techniques have been devised in the past to obtain the approximate values based on a perturbation expansion in some small parameter. The Lindstedt–Poincaré method [6] and the multiple-scale method [7] are some of the techniques that have been used for this kind of problems. For solving the integral in (3) it is appropriate to make a change of the variable putting [8]

$$x = \frac{x_+ + x_-}{2} + \frac{x_+ - x_-}{2}\cos\theta$$

which transforms the above integral on a new one over the interval $[0, \pi]$. In general, this integral must be solved numerically.