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Computing survival probabilities based on stochastic differential models

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HIGHLIGHTS

- We propose a very efficient method to compute survival probabilities.
- We combine polynomial differential quadrature with high-order time-stepping.
- We consider a reduced-form model and a structural model that arise from finance and insurance.
- The method is model independent and could also be extended to other stochastic processes.
- Numerical comparison with other recent approaches is provided.

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ABSTRACT

We develop a new numerical method to compute survival probabilities based on stochastic differential models, a matter of great importance in several areas of science, such as finance, biology, medicine and geophysics. This novel approach is based on polynomial differential quadrature, which is combined with a high-order time discretization scheme. Numerical experiments are presented showing that the proposed method performs extremely well and is more efficient than the approaches recently developed in Costabile et al. (2013) and Guarin et al. (2011).

1. Introduction

Survival probabilities play an important role in several areas of science such as geophysics (see [1]), medicine (see [2–4]), biology (see [5]). Nevertheless, they have the most relevant applications in finance, where they are largely employed for credit risk modeling (see [6–17]) and for insurance risk modeling (see [18–31]).

Survival probabilities are mainly computed using two different kinds of approaches, namely *reduced-form* models and *structural* models. According to the reduced-form models (see, e.g., [32,19,33,10,1,34]), the so-called time of death (that can be the default time of a firm, or the time at which an individual dies, or the time at which a volcano erupts, or the time at which two molecules cease to communicate, or …) is specified as the first jump of a Poisson process, whose intensity is often considered to be stochastic. The structural approach (see, e.g., [7,20–22,24–28,15,30,31,17]), instead, aims at incorporating the dynamics of some variable on which the survival probability depends (for instance, when modeling the default of a firm, such a variable can be the value of the firm's assets). In particular, the time of death is usually specified as the first time when the structural variable, which is assumed to be stochastic, falls below a given threshold level (barrier).

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Quite often, reduced-form models and structural models do not have closed-form solution and thus require some kind of approximation. In particular, a numerical method to solve reduced-form models with one and two stochastic factors has recently been proposed by Guarin, Liu and Ng [10]. Nevertheless, such an approach, which is based on radial basis functions, is not extremely efficient, as it takes about one second to compute survival probabilities with an error (in the maximum norm) of order 10^{-3}.

Instead, as far as the numerical approximation of structural models is concerned, semi-analytical formulae and probabilistic methods are presented in [20,21,24–28,30,31]. Nevertheless, all these approaches are tailored on the specific model to which they are applied and as such they are not very flexible to be extended to more general cases. A remarkable contribution to the numerical simulation of structural models can be found in a recent paper by Costabile, Massabò and Russo [22], where a lattice-based multinomial approximation is proposed which offers several advantages: it is very general and flexible, it is relatively simple to implement, and it provides a direct recursive calculation of the survival probability. However, to obtain survival probabilities with a relative error of order 10^{-3} one may need up to 90 s (on a standard computer), and so, from the computational standpoint, the approach proposed in [22] leaves space for further improvements.

In the present manuscript we develop a numerical method to compute survival probabilities based on polynomial differential quadrature (PDQ). This technique, which has been originally introduced in [35], is very versatile and has also the advantage that a relatively small number of nodes, say some tens in each coordinate direction, is sufficient to reach very high levels of accuracy (see [36,35,37]). In particular, the PDQ approach is used to approximate the spatial derivatives, whereas the time discretization is performed by means of a high-order finite difference scheme, so that we can employ a computational mesh that is coarse along both the space and the time variables.

The above numerical approach is tested on two different problems, namely the reduced-form model presented in [10] and a structural model considered in [22]. As far as the reduced-form model is concerned, the results obtained are excellent: on a standard computer the survival probabilities are calculated with an error (in the maximum norm) of order 10^{-4} in only few hundredths of a second. Moreover, 20 nodes (in each spatial direction) are sufficient to obtain an error (in the maximum norm) of order 10^{-5}, and on the overall the novel PDQ scheme is hundreds of times more efficient than the radial basis function approach employed in [10]. Instead, when it comes to the numerical simulation of the structural model, the PDQ method reveals to be less accurate than in the reduced-form case (to obtain errors of order 10^{-5} the use of approximately 80 nodes is required), which is due to the fact that the solution to be approximated experiences strong spatial gradients in the region close to the barrier. Nevertheless, the results obtained are still very satisfactory. In fact, the survival probability is computed with an error (in the maximum norm) of order 10^{-5} in only few hundredths of a second. Furthermore, as shown in Section 5.2, the proposed PDQ method reveals to be significantly more efficient than the multinomial lattice developed in [22].

Finally, it is worth noticing that the novel approach is very general and versatile, and thus could also be applied to models based on stochastic processes other than those considered in this manuscript.

The remainder of the paper is organized as follows: in Section 2 the reduced-form model presented in [10] is briefly shown; in Section 3 the PDQ method is described by applying it to the reduced-form model outlined in Section 2; in Section 4 an extension of the proposed PDQ approach to a structural model considered in [22] is developed; in Section 5 the most relevant results obtained for the two models are presented and discussed; finally, in Section 6 some conclusions are drawn.

2. Computing survival probabilities based on a reduced-form model

In this section we consider the problem of computing survival probabilities based on the reduced-form approach. Specifically, we focus our attention on a reduced-form model that has recently been proposed in [10], according to which the default intensity is specified as a two-factor Exponential-Vasicek (EV) process. Note that the EV process are frequently used to describe default intensities (see [38,33,10,13,14,16]), as they offer several advantages: they remain strictly positive, they are mean reverting and stationary in the long run and they have positive skewness and fat tails [39].

Let τ denote the time of the first jump of a (doubly stochastic) Poisson process [33] whose intensity, hereafter denoted by λ, follows the two-factor EV dynamics [10]:

$$\lambda_s = e^{x_s} + e^{y_s}, \tag{1}$$

$$dx_s = \alpha(\ln\theta - x_s)ds + \sigma dW_s^{(x)}, \tag{2}$$

$$dy_s = \eta(\ln\mu - y_s)ds + \upsilon dW_s^{(y)}, \tag{3}$$

where $W^{(x)}$ and $W^{(y)}$ are standard Brownian motions with constant correlation ρ and $\alpha, \theta, \sigma, \eta, \mu, \upsilon$ are constant parameters.

Let T denote a fixed time, and let $P(x, y, s; \tau \geq T)$ denote the probability of τ being greater than or equal to T given $\tau \geq s$ and $(x_s, y_s) = (x, y)$, with $T \geq s$. The following formula holds (see [40]):

$$P(x, y, s; \tau \geq T) = E\left[e^{-\int_s^T e^{x_z} + e^{y_z} dz} \,\middle|\, (x_s, y_s) = (x, y) \right]. \tag{4}$$

Noting that the processes (1)–(3) are time-homogeneous, so that $P(x, y, s; \tau \geq T) = P(x, y, 0; \tau \geq T - s)$, let us apply the change of variables $t = T - s$, and let us define $Q(x, y, t) = P(x, y, 0; \tau \geq t)$. By using the Feynman–Kac theorem