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Numerical proper reparametrization of parametric plane curves☆

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HIGHLIGHTS

- We introduce the problem of numerical improper parametrization of plane curves.
- The approximate improper index is firstly defined in this paper.
- A numerical algorithm is designed to find the approximate proper reparametrization.
- The relationship between the numerical curve and its reparametrization is well studied.

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ABSTRACT

We present an algorithm for reparametrizing algebraic plane curves from a numerical point of view. More precisely, given a tolerance $\epsilon > 0$ and a rational parametrization $\mathcal{P}$ of a plane curve $\mathcal{C}$ with perturbed float coefficients, we present an algorithm that computes a parametrization $\mathcal{Q}$ of a new plane curve $\mathcal{D}$ such that $\mathcal{Q}$ is an ϵ *–proper reparametrization* of $\mathcal{D}$. In addition, the error bound is carefully discussed and we present a formula that measures the "closeness" between the input curve $\mathcal{C}$ and the output curve $\mathcal{D}$.

1. Introduction

Let $\mathcal{P}(t)$ be a rational affine parametrization of an algebraic plane curve $\mathcal{C}$ over the complex field $\mathbb{C}$. Associated with $\mathcal{P}(t)$, we have the rational map $\phi_{\mathcal{P}} : \mathbb{C} \to \mathcal{C}; t \to \mathcal{P}(t)$, where $\phi_{\mathcal{P}}(\mathbb{C}) \subset \mathcal{C}$ is dense. $\phi_{\mathcal{P}}$ is a birational map if $\mathcal{P}$ is proper. That is, except for a finite number of points, for almost every point $p \in \mathcal{C}$, there is exactly one parameter value $t_0 \in \mathbb{C}$ such that $\mathcal{P}(t_0) = p$. Geometrically, $\mathcal{P}$ proper means that $\mathcal{P}$ traces the curve once. If $\mathcal{P}$ is not proper, there is more than one parameter value corresponding to a generic point on $\mathcal{C}$. Lüroth's Theorem shows constructively that it is always possible to reparametrize an improperly parametrized curve to a proper one.

Proper parameterizations are crucial to many practical problems in computer aided geometric design (CAGD), such as visualization (see [1,2]). Particularly, proper parameterizations ensure the validity of the resultant technique in the implicitization problem (see [3–6]). Therefore, the proper reparametrization problem has received extensive research (see [7–11] for examples).

The problem of proper reparametrization for curves has widely been discussed from the symbolic point of view. More precisely, *given the field of complex numbers* $\mathbb{C}$, *and a rational parametrization* $\mathcal{P}(t) \in \mathbb{C}(t)^2$ *of an algebraic plane curve* $\mathcal{C}$ *with*

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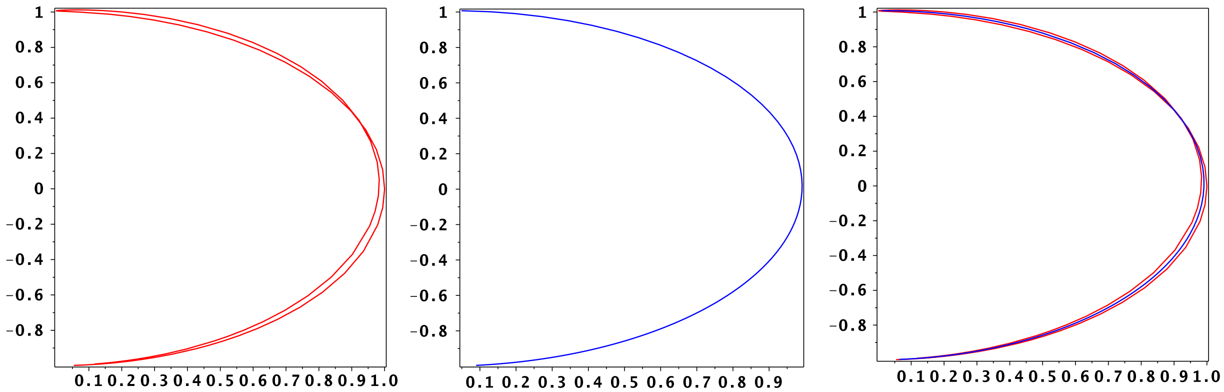

Fig. 1. Input curve $\mathcal{C}$ (left), curve $\mathcal{D}$ (center), curves $\mathcal{C}$ and $\mathcal{D}$ (right).

exact coefficients, one computes a rational proper parametrization $\mathcal{Q}(t) \in \mathbb{C}(t)^2$ *of* $\mathcal{C}$, *and a rational function* $R(t) \in \mathbb{C}(t)\backslash\mathbb{C}$ *such that* $\mathcal{P}(t) = (\mathcal{Q} \circ R)(t)$. Nevertheless, in many practical applications, symbolic (or exact) approaches tend to be insufficient, since in practice object data are usually given approximately. As a consequence, hybrid symbolic–numerical algorithms have stepped onto stage.

Briefly speaking, given a tolerance $\epsilon > 0$, and an irreducible affine algebraic plane curve $\mathcal{C}$ defined by a parametrization $\mathcal{P}$ with perturbed float coefficients that is "*nearly improper*" (i.e. improper within the tolerance ϵ), one looks for a rational curve $\mathcal{D}$ defined by a parametrization $\mathcal{Q}$, such that $\mathcal{Q}$ is proper and almost all points of the rational curve $\mathcal{D}$ are in the "*vicinity*" of $\mathcal{C}$. The notion of vicinity can be illustrated by the offset region restricted by the external and internal offset to $\mathcal{C}$ at distance ϵ (see Section 4 for details). Therefore, the problem reduces to find a properly parametrized curve $\mathcal{D}$ that lies within the offset region of $\mathcal{C}$. For instance, assume that we are given a tolerance $\epsilon = 0.2$, and a curve $\mathcal{C}$ defined by the parametrization

$$\mathcal{P}(t) = \left(\frac{1.999t^2 + 3.999t + 2.005 - 0.003t^4 + 0.001t^3}{2.005 + 0.998t^4 + 4.002t^3 + 6.004t^2 + 3.997t}, \frac{0.001 - 0.998t^4 - 4.003t^3 - 5.996t^2 - 4.005t}{2.005 + 0.998t^4 + 4.002t^3 + 6.004t^2 + 3.997t} \right).$$

One may check that $\mathcal{P}$ is proper from the symbolic point of view; but it is nearly improper (numerically speaking), since for almost all points $p := \mathcal{P}(s_0) \in \mathcal{C}$, $s_0 \in \mathbb{C}$, there exist two values of the parameter t, given by the approximate roots of the equation $0.4901606943t^2 + 0.2393271335\ 10^{-8}(2202769s_0 + 417838122)t - 0.4954325182s_0^2 - s_0 = 0$, such that $\mathcal{P}(t)$ is "*almost equal*" to $\mathcal{P}(s_0)$. Our method provides an ϵ-*proper reparametrization* of $\mathcal{D}$

$$\mathcal{Q}(t) = \left(\frac{-0.00139214373770521\,t^2 - 0.455587113115768\,t + 0.230804565878748}{0.472790306463932\,t^2 - 0.475516806696674\,t + 0.233345983511073}, \right.$$
$$\left. \frac{-0.472791477433681\,t^2 + 0.473001908925789\,t - 0.00421763512489261}{0.472790306463932\,t^2 - 0.475516806696674\,t + 0.233345983511073} \right).$$

In Fig. 1, one may check that $\mathcal{C}$ and $\mathcal{D}$ are "*close*".

To relate the tolerance with the vicinity region, one can either approach from analyzing locally the condition number of the implicit equations (see [12]), estimating the Hausdorff distance (see e.g. [13,14]) or studying whether for almost every point p on the original curve, there exists a point q on the output curve such that the distance of p and q is significantly smaller than the tolerance (see e.g. [15,16]). The error analysis we present in this paper will be based on the third approach, and we shall derive upper bounds for the distance of the offset region.

Approximate algorithms have been developed for many applied numerical topics, such as computing approximate greatest common divisor (gcd) [17–21], finding zeros of multivariate systems [19,22], factoring polynomials [23,24], etc. In addition, computing approximate parametrizations for algebraic curves and surfaces has been investigated. For instance, in [25], the authors construct a C^1-continuous piecewise (m, n) rational ϵ-approximation of a real algebraic plane curve. Using the Weierstrass Preparation Theorem, Newton power series factorizations, and modified rational Padé approximations, the authors construct a locally approximate rational parametric representations for all real branches of the given algebraic plane curve. In [26], a novel approach for computing an approximate parametrization of a whole closed space algebraic curve from a small number of approximating arcs is presented. [27] proposes an algorithm that subdivides the given curve into arcs, and then approximates the arcs with curves parametrized by rational functions of low degree. In [28], a method for computing an approximate parametrization of a planar algebraic curve by a rational Bézier (spline) curve is described. The approach is based on the minimization of a suitable non-linear objective function, which takes into account both the distance from the curve and the positivity of the weight function (i.e., the numerator of the rational parametric representation). In [29], a method of approximating a segment of the intersection curve of two implicitly defined surfaces by a rational parametric curve is presented. The method includes predictor and corrector steps. The corrector step is formulated