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Error-driven dynamical *hp*-meshes with the Discontinuous Galerkin Method for three-dimensional wave propagation problems

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ABSTRACT

An *hp*-adaptive Discontinuous Galerkin Method for electromagnetic wave propagation phenomena in the time domain is proposed. The method is highly efficient and allows for the first time the adaptive full-wave simulation of large, time-dependent problems in three-dimensional space. Refinement is performed anisotropically in the approximation order p and the mesh step size h regardless of the resulting level of hanging nodes. For guiding the adaptation process a variant of the concept of reference solutions with largely reduced computational costs is proposed. The computational mesh is adapted such that a given error tolerance is respected throughout the entire time-domain simulation.

1. Introduction

In this article, we are concerned with adaptively solving the Maxwell equations for electromagnetic fields with arbitrary time dependence in a three-dimensional domain such that a prescribed error tolerance is respected. In order to achieve this goal the Discontinuous Galerkin Method (DGM) [1,2] is applied on anisotropic *hp*-meshes, which dynamically and autonomously adapt as the electromagnetic fields evolve. The mesh refinement is driven by a robust local error estimate based on a modification of the so-called method of reference solutions [3,4] with largely reduced numerical costs.

The DG method has gained wide acceptance as a high order numerical method, which is very well suited for time-domain problems. It combines the usually opposing key features of high order accuracy and flexibility. In particular, the method can easily deal with meshes containing hanging nodes, which makes it particularly well suited for *hp*-adaptivity. There is a well established body of literature on the DGM for various types of problems available. It has been thoroughly investigated by several research groups (see e.g. [5–7] and references therein). Concerning Maxwell's equations in the time domain, the DGM has been studied in particular in [7–10]. The latter two make use of hexahedral meshes, which allow for a computationally more efficient implementation [11].

The simplest approach to adapted grids consists of static *a priori h*-refinement around edges and corners, i.e., the possible locations of field singularities [12]. While this approach mitigates negative effects of fields singularities on the global solution accuracy, the level of refinement to be applied for achieving a certain accuracy is unknown. Moreover, edges and corners require no mesh refinement while there is no field, for instance, before illumination by a wave or after scattering took place. It also remains unclear how to choose polynomial orders in the remaining mesh. For these reasons our focus is on *hp*-adaptivity based on error estimations of the time-dependent solution.

Mesh refinement and specifically *hp*-adaptation has received considerable and continuous attention. The first published work on *h*-, *p*- and *hp*-adaptivity within the DG framework is presumably [13], where the authors considered linear scalar hyperbolic conservation laws in two-dimensional space. Hyperbolic problems have also been addressed, e.g., by

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Flaherty, Shephard and co-workers who considered two-dimensional problems in [14,15] as well as three-dimensional settings with pure *h*-refinement in [16,17]. A large number of contributions has been authored by Houston and various co-workers. They present a number of approaches to adaptivity and deal with first-order hyperbolic problems in [18,19], using adjoint solutions [19,20] or estimating errors in an energy norm [21,22]. The contributions have a clear focus on the rigorous derivation of error estimates and error bounds. Applications are limited to one or two space dimensions. Recently, Solin and co-workers published papers, where they apply dynamical *hp*-meshes for various coupled problems including electromagnetics in two space dimensions [23–25]. They employ the concept of reference solutions for controlling mesh adaptivity and perform refinements, which are fully anisotropic in both mesh parameters h and p. The application of reference solutions in their original form is numerically very expensive. In [25] it is stated that the solution of large three-dimensional problems would require distributed parallel computing.

In this paper, we propose a modification of the concept of reference solutions with drastically reduced numerical costs, which makes such simulations feasible. At the same time the key advantages are maintained, in particular its robustness and the independence of a particular set of underlying partial differential equations. The increased efficiency comes at the price of losing some sharpness in the error estimate. Like the original formulation, the proposed algorithm is entirely devoid of tuning parameters, and it reduces the true approximation error, i.e., it is not based on residuals or heuristic measures such as steep gradients. The adaptation can be performed in four major modes: isotropic in h and p, anisotropic in one of h or p, and fully anisotropic in h and p. Unconstrained refinement in h is possible because we allow for high level hanging nodes. The number of degrees of freedom (DoF) in a discretization will usually decrease from the former to the latter mode, while the computational load for finding the adapted mesh increases. However, we will show below that great savings in both the number of DoF and computational time can be achieved by using fully anisotropic adaptivity.

The remainder of this article is organized as follows. In Section 2 the notation and Finite Element Spaces (FES) are introduced, which are applied for obtaining a weak DG formulation of Maxwell's equations. Section 3 is devoted to the mesh refinement algorithm. First the individual steps, which constitute an adaptive algorithm are discussed. They are error estimation, element marking, the *h*–*p*-decision and the actual mesh adaptation. For each step a brief description with a review of the state of the art is provided, before we proceed with the details of our realization of each step in Sections 3.1–3.5. Examples are presented in Section 4, which include a waveguide and an antenna radiation problem. Section 5 summarizes the findings and concludes the article.

2. Discretization of Maxwell's equations

In the following we assume resting, heterogeneous, linear, isotropic, non-dispersive and time-independent materials. Then, the magnetic permeability, μ, and dielectric permittivity, ϵ, are scalar values depending on the spatial position only. Under these assumptions Maxwell's equations read

$$\nabla \times \mathbf{E}(\mathbf{x}, t) = -\mu(\mathbf{x}) \frac{\partial}{\partial t} \mathbf{H}(\mathbf{x}, t), \tag{1}$$

$$\nabla \times \mathbf{H}(\mathbf{x}, t) = \epsilon(\mathbf{x}) \frac{\partial}{\partial t} \mathbf{E}(\mathbf{x}, t) + \mathbf{J}(\mathbf{x}, t), \tag{2}$$

with the spatial variable $\mathbf{x} \in \Omega \subset \mathbb{R}^3$ and the temporal variable $t \in [t_0, T] \subset \mathbb{R}$ subject to boundary conditions specified at the domain boundary $\partial\Omega$ and initial conditions specified at time t_0. The electric and magnetic field vectors are denoted by $\mathbf{E}$ and $\mathbf{H}$, $\mathbf{J}$ denotes the electric current density.

Discretizations of Maxwell's equations using the Discontinuous Galerkin Method have been obtained among others in [7–10]. We will follow the framework and notation described in our previous work [26], which makes use of hexahedral meshes and modal basis functions as introduced in [10].

2.1. Notation

We denote by $\mathcal{T}_h$ a tessellation of the domain of interest Ω composed from non-overlapping hexahedra $\mathcal{T}_i$ such that $\mathcal{T}_h = \bigcup_{i=1}^{N} \mathcal{T}_i$ covers Ω. The tessellation is required to be derivable from a regular root tessellation $\mathcal{T}_0$ by means of element bisections. However, we do not demand the resulting tessellation to be regular, i.e., we allow for hanging nodes and specifically for high level hanging nodes. The number of bisections performed for obtaining element $\mathcal{T}_i$ is denoted by L_i in the isotropic and $L_{d,i}$ in the anisotropic case where d corresponds to any of the spatial coordinates $\{x, y, z\}$. We call the intersection of two neighboring elements $\mathcal{T}_i \cap \mathcal{T}_k$ their interface $\mathcal{I}_{ik}$. In non-conformingly refined meshes, every face $\mathcal{F}_j$ of a hexahedral element may be partitioned into several interfaces depending on the number of neighbors K such that $\mathcal{F}_j = \bigcup_{k=1}^{K} \mathcal{I}_{ik}$. This is an important difference to most other works including [7,9,27], which require one-to-one neighborhood relations. The (inter-)face orientation is described by the outward pointing unitary normal $\mathbf{n}_j$. The union of all faces is denoted by $\mathcal{F}$. The volume and edge length measures of element i are denoted by $|\mathcal{T}_i|$ and $|\mathcal{T}_{d,i}|$.