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A hybrid level set/front tracking approach for finite element simulations of two-phase flows

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ABSTRACT

In this paper we give details on the numerical realization of a new finite element method for the simulation of two-phase flows which was recently introduced in Basting and Weismann (2013). The main ingredient is a hybrid representation of the interface between the fluid phases: An implicit description of the interface is given by a level set function and an explicit representation is obtained from aligning edges of the computational mesh to the implicitly described interface. This step is done by a black-box optimization based mesh smoothing approach which does not change the topology of the mesh while guaranteeing optimal mesh quality. Furthermore, we make use of quadratic isoparametric elements to increase the approximation quality of the discrete interface.

Due to the alignment, discontinuities of the solution variables (pressure) can be captured accurately, while a variational treatment of the curvature allows for a precise approximation of surface tension. We present our time discretization scheme for the coupled Navier–Stokes/level set equations, and discuss our space discretization based on the so called subspace projection method (SPM) to account for discontinuities across the interface.

We present two numerical examples for which reference solutions exist. We consider the oscillation of a single droplet and provide our results for an established two-phase flow benchmark problem.

1. Introduction

When it comes to numerical methods for flow problems with free interfaces, the representation of the interface is a key issue. Consequently, over the years, many approaches have been proposed. Most of these approaches may be classified as either *interface capturing* or *interface tracking* methods.

In *interface capturing* methods, the interface is represented implicitly by an additional function, for instance as a distance function in the level set method [1,2] or by means of a volume fraction in VOF methods [3]. Commonly, these methods are defined on structured meshes. Due to the implicit representation of the interface, these methods are especially powerful when strong deformations of the interface occur. However, if interface forces such as surface tension play an essential role, special care has to be taken with regard to its discretization. Furthermore, since the computational mesh is in general not aligned with the interface, solution properties such as discontinuities of the pressure across the interface are difficult to capture.

On the other hand, in *interface tracking* methods, the interface is discretized explicitly. This can be achieved by additional markers which are transported by the flow field, or by a separate interface mesh. A special class of interface tracking

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methods is obtained from aligning the computational mesh with the discrete interface. Usually, in these *aligned interface methods*, equations are formulated in arbitrary Lagrangian–Eulerian (ALE) coordinates [4] which allows for the movement of the computational mesh with the interface. In this case, the aforementioned problems of surface tension evaluation and representation of discontinuities of the pressure can be treated very easily: due to the alignment, a discrete representation of the interface is always at hand in terms of edges of the computational mesh, and finite element spaces taking discontinuities into account may be realized quite easily. However, the movement may lead to degeneration of the computational mesh. Although techniques such as remeshing or special extension operators allow to deal with more complex situations (see for instance [5,6] for numerical studies on different mesh moving strategies for problems with large deformations), these methods are usually applied when deformations of the interface can be expected to be "mild".

A hybrid approach which aims at combining interface capturing and interface tracking methods to achieve enhanced geometrical flexibility while retaining the benefits of aligned mesh methods was introduced in [7] and applied to particulate flows in [8]. The main idea is to use a level set representation of the interface while aligning the computational mesh with the zero level set in each time step. This is achieved in an automatic way by a black-box mesh optimization approach. The interface is always approximated by certain edges of the mesh (which are not specified a priori as in "classical" aligned interface methods). In this paper, we review this approach and give details on the time and space discretization of the method presented in [7]: We discuss a splitting of the coupled Navier–Stokes/level set equations in time, time discretization of the resulting subproblems and show how a finite element space which is able to capture the discontinuity of the pressure across the interface can be realized using a discrete projection (the subspace projection method), which was introduced in [9–11].

We present two numerical examples to demonstrate the benefits but also the limitations of our proposed approach.

2. Mathematical model

We consider the behavior of two immiscible, incompressible Newtonian fluids modeled by the incompressible Navier–Stokes equations. More precisely, we assume to have one time independent domain $\Omega \subset \mathbb{R}^2$ occupied by two time dependent fluid domains $\Omega_1(t), \Omega_2(t)$ which are separated by a sharp interface $\Gamma(t)$, i.e. $\bar{\Omega} = \bar{\Omega}_1(t) \cup \bar{\Omega}_2(t)$, $\Omega_1(t) \cap \Omega_2(t) = \emptyset$ and $\Gamma(t) = \bar{\Omega}_1(t) \cap \bar{\Omega}_2(t)$ for time instants $t \in [0, T]$. In each domain $\Omega_i(t)$, we require the fluid to have constant density ρ_i and viscosity μ_i.

The governing equations in the bulk read in dimensionless form

$$\left.\begin{aligned} &\Lambda_i \left(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u}\right) - \nabla\cdot\boldsymbol{\sigma}_i = \Lambda_i \mathbf{f} \\ &\nabla\cdot\mathbf{u} = 0 \end{aligned}\right\} \quad \text{in } \Omega_i(t), \tag{1}$$
$$\mathbf{u} = 0 \quad \text{on } \partial\Omega,$$

where $\mathbf{u}$ denotes velocity, p pressure, $\mathbf{f}$ denotes the vector of external forces and

$$\Lambda_i = \frac{\rho_i}{\rho_c}, \qquad \mathrm{Re}_i = \frac{\rho_i U L}{\mu_i}, \qquad \boldsymbol{\sigma}_i = \frac{1}{\mathrm{Re}_i}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right) - p\mathcal{I}$$

denote density ratio (with reference density ρ_c), Reynolds number and stress tensor for each domain. U denotes a characteristic velocity and L a characteristic length scale.

If we denote the constant surface tension coefficient by σ and the curvature of Γ by κ, the capillary boundary condition on Γ is given by

$$[\![\boldsymbol{\sigma}(\mathbf{u}, p)]\!] = \frac{1}{\mathrm{We}}\kappa\mathbf{n} \quad \text{on } \Gamma(t) \tag{2}$$

with Weber number $\mathrm{We} = \frac{\rho_c U^2 L}{\sigma}$. The movement of the interface Γ is prescribed by the kinematic boundary condition

$$V_\Gamma = \mathbf{u}\cdot\mathbf{n} \quad \text{on } \Gamma(t). \tag{3}$$

In this paper, we make use of the arbitrary Lagrangian–Eulerian (ALE) formulation [4] of the mathematical model to follow the movement of the fluid interface. To this end, we consider a fixed reference domain $\hat{\Omega} \subset \mathbb{R}^2$ whose boundary coincides with the boundary of Ω, i.e. $\partial\hat{\Omega} = \partial\Omega(t)\ \forall t$. We assume to have a smooth mapping

$$\boldsymbol{\xi} : [0, T] \times \hat{\Omega} \to \mathbb{R}^2,$$
$$\boldsymbol{\xi}(t, \hat{\Omega}) = \Omega(t) \quad \text{for all } t \in [0, T].$$

For each time instant $t \in [0, T]$, we assume $\boldsymbol{\xi}$ to be a homeomorphism. The velocity of the domain $\mathbf{w}$ is defined as

$$\begin{aligned} &\mathbf{w}(t, \cdot) : \Omega(t) \to \mathbb{R}^2, \\ &\mathbf{w}(t, \cdot) = \partial_t \boldsymbol{\xi}(t, \boldsymbol{\xi}(t, \cdot)^{-1}). \end{aligned} \tag{4}$$

For any sufficiently smooth function $F : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ we may define the ALE time derivative of F as

$$\hat{\partial}_t F(t, \mathbf{x}) := \partial_t F(t, \boldsymbol{\xi}(t, \hat{\mathbf{x}})) = \partial_t F(t, \mathbf{x}) + \mathbf{w}(t, \mathbf{x})\cdot\nabla F(t, \mathbf{x}) \tag{5}$$

for $\mathbf{x} = \boldsymbol{\xi}(t, \hat{\mathbf{x}}),\ \hat{\mathbf{x}} \in \hat{\Omega}$.