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A certain class of starlike log-harmonic mappings

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ABSTRACT

In this paper we investigate some properties of log-harmonic starlike mappings. For this aim we use the subordination principle or Lindelof Principle (Lewandowski (1961) [7]).

1. Introduction

Let $H(D)$ be the linear space of all analytic functions defined on the open unit disc $\mathbb{D} = \left\{z \mid |z| < 1\right\}$, and let $B = \left\{w(z) \mid |w(z)| < 1\right\}$ for all $z \in \mathbb{D}$.

A function $f : \mathbb{D} \to C$ is said to be log-harmonic on $\mathbb{D}$ if there is a $w(z) \in B$ such that f is a non-constant solution of the non-linear elliptic partial differential equation

$$\frac{\overline{f}_{\overline{z}}}{\overline{f}} = w(z)\frac{f_z}{f} \tag{1.1}$$

where the second dilatation function $w(z) \in \mathbb{D}$ is such that $|w(z)| < 1$ for all $z \in \mathbb{D}$. It has been shown that if f is a non-vanishing log-harmonic mapping, then f can be expressed on

$$f = h(z)\overline{g(z)} \tag{1.2}$$

where $h(z)$ and $g(z)$ are analytic functions in $\mathbb{D}$. On the other hand, if f vanishes at $z = 0$, but is not identically zero, then f admits the following representation:

$$f = z\,|z|^{2\beta}\, h(z)\overline{g(z)} \tag{1.3}$$

where $\operatorname{Re}\beta > -\frac{1}{2}$ and $h(z)$ and $g(z)$ are analytic functions in $\mathbb{D}$, $g(0) = 1$ and $h(0) \neq 0$ (see [1]). Univalent log-harmonic mappings have been studied extensively [1–5]. We note that the class of log-harmonic mappings is denoted by S_{LH}.

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Let $f(z) = z\,|z|^{2\beta}\,h(z)\overline{g(z)}$ log-harmonic be a univalent mapping. We say that f is a starlike log-harmonic mapping if

$$\frac{\partial \mathrm{Arg} f(re^{i\theta})}{\partial\theta} = \mathrm{Re}\frac{zf_z - \overline{z}f_{\overline{z}}}{f} > 0 \tag{1.4}$$

for all $z \in \mathbb{D}$. Denote by ST_{LH} the set of all starlike log-harmonic mappings. Finally, let Ω be the family of functions $\phi(z)$ which are regular in $\mathbb{D}$ and satisfy the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$.

Denote by P the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ regular in $\mathbb{D}$ such that $p(z)$ is in P if and only if

$$p(z) = \frac{1+\phi(z)}{1-\phi(z)} \tag{1.5}$$

for some function $\phi(z) \in \Omega$ and for every $z \in \mathbb{D}$.

Let S^* denote the family of functions $s(z) = z + c_2 z^2 + c_3 z^3 + \cdots$ regular in $\mathbb{D}$, and such that $s(z)$ is in S^* if and only if

$$\mathrm{Re}\left(z\frac{s'(z)}{s(z)}\right) > 0 \Leftrightarrow z\frac{s'(z)}{s(z)} = p(z) \tag{1.6}$$

for some function $p(z) \in P$ and every $z \in \mathbb{D}$.

Let $s_1(z)$ and $s_2(z)$ be analytic functions in $\mathbb{D}$ with $S_1(0) = S_2(0)$. We say that $s_1(z)$ is subordinate to $s_2(z)$ and denote it by $s_1(z) \prec s_2(z)$ if $s_1(z) = s_2(\phi(z))$ for some function $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$. If $s_2(z)$ is univalent, then $S_1(\mathbb{D}_r) \subset S_1(\mathbb{D}_r)$, $\mathbb{D}_r = z|\,|z| < r$, $0 < r < 1$ (subordination principle or Lindelof Principle [5]).

Lemma 1.1 ([6]). *Let $\phi(z)$ be regular in the unit disc $\mathbb{D}$, with $\phi(0) = 0$. Then if $|\phi(z)|$ attains its maximum value on the disc $|z| = r$ at the point z_1, one has $z_1\phi'(z_1) = k\phi(z_1)$ for some $k \geq 1$.*

2. Main results

Theorem 2.1. *Let $f(z) = z\,|z|^{2\beta}\,h(z)\overline{g(z)}$ be an element of ST_{LH}, then*

$$f \in ST_{LH} \Leftrightarrow \left(z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)}\right) \prec \frac{2z}{1-z}.$$

Proof. Let $f(z) = z\,|z|^{2\beta}\,h(z)\overline{g(z)}$ be an element of ST_{LH}, and we consider the Riemann branch $1^{2\beta} = 1$. Then

$$\begin{cases} \dfrac{zf_z}{f} = 1 + \beta + z\dfrac{h'(z)}{h(z)}, \\ \dfrac{\overline{z}f_{\overline{z}}}{f} = \beta + \overline{z}\overline{\dfrac{g'(z)}{g(z)}}, \end{cases} \Rightarrow \frac{zf_z - \overline{z}f_{\overline{z}}}{f} = 1 + z\frac{h'(z)}{h(z)} - \overline{z}\overline{\frac{g'(z)}{g(z)}}. \tag{2.1}$$

Thus we have

$$\begin{aligned} &0 < \mathrm{Re}\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\right) = \mathrm{Re}\left(1 + z\frac{h'(z)}{h(z)} - \overline{z}\overline{\frac{g'(z)}{g(z)}}\right) = \mathrm{Re}\left(1 + z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)}\right) \\ &\Leftrightarrow 1 + z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)} = p(z) = \frac{1+\phi(z)}{1-\phi(z)} \Leftrightarrow z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)} = \frac{2\phi(z)}{1-\phi(z)} \\ &\Leftrightarrow z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)} \prec \frac{2z}{1-z} \end{aligned}$$

where $p(z) \in P$, $\phi(z) \in \Omega$. □

Corollary 2.2. *Let $f = z\,|z|^{2\beta}\,h(z)\overline{g(z)}$ be an element of ST_{LH}, then*

$$\left|\left(\frac{h(z)}{g(z)}\right)^2 - 1\right| < 1. \tag{2.2}$$