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Riemann solutions for spacetime discontinuous Galerkin methods

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ABSTRACT

Spacetime discontinuous Galerkin finite element methods (cf. Abedi et al. (2013), Abedi et al. (2010), Miller and Haber (2008)) rely on 'target fluxes' on element boundaries that are computed via local one-dimensional Riemann solutions in the direction normal to the element face. In this work, we provide details of converting a space–time flux expressed in differential forms into a standard one-dimensional Riemann problem on the element interface. We then demonstrate a generalized solution procedure for linearized hyperbolic systems based on diagonalization of the governing system of partial differential equations. The generalized procedure is particularly useful for the implementation aspects of coupled multi-physics applications. We show that source terms do not influence the Riemann solution in the spacetime setting. We provide details for implementation of coordinate transformations and Riemann solutions. Exact Riemann solutions for some linearized systems of equations are provided as examples, including an exact, semi-analytic Riemann solution for generalized thermoelasticity with one relaxation time.

1. Introduction

Riemann solvers or approximate Riemann solvers are commonly used in control volume based numerical methods, e.g. finite volume methods and discontinuous Galerkin methods [1–3]. Spacetime discontinuous Galerkin (SDG) methods, as described in [4–14], are one such numerical method where Riemann solutions (or Riemann fluxes) are needed on inter-elemental boundaries. In this paper, we provide the details of the Riemann solution process as we have used it in [4–9]. A key component of the procedure is transforming the differential forms formulation into a local coordinate system in order to solve the standard Riemann problem in the normal direction. The Riemann solution procedure itself is not novel; the basic procedure can be found in articles [15] or textbooks [1,16]. The usefulness of our current exposition lies in clarifying the transition from differential forms in spacetime to a one dimensional Riemann problem, and calculating the exact Riemann flux. The general procedure we develop can be applied to systems of hyperbolic equations, and it is not restricted to SDG methods. In addition, our semi-analytic solution structure is particularly useful for the derivation and implementation of the Riemann solutions for complicated multiphysics problems, as shown in Section 4.3 on generalized thermoelasticity.

We use differential forms and the exterior calculus on manifolds to formulate systems of hyperbolic equations and express fluxes across spacetime interfaces with arbitrary orientation. This approach yields a very concise and elegant structure for various identities such as the Stokes theorem. More importantly, it eliminates problems pertained to orthogonality and the definition of magnitude and normal vectors in classical mechanics due to the absence of an objective definition for spacetime normal vectors. While many of the mathematical statements necessary can be made with 'standard'

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tensor calculus notation, we find that differential forms in spacetime are extremely useful in identifying the correct spacetime fluxes, restrictions on arbitrarily oriented boundaries, and dual quantities. Differential forms for finite element methods are not quite standard, but their usage is increasing due to their compact notation and elucidation of physical and mathematical properties (see, e.g., [17–20]).

We follow by discussing the restrictions of differential forms to element faces and basic coordinate transformations. We then state the Riemann problem for a general linear hyperbolic system and demonstrate the solution construction on arbitrarily oriented boundaries. Source terms in the differential system are shown to have no effect on the Riemann solution (in the SDG context). We also provide examples from linearized elastodynamics, non-Fourier heat conduction, and a form of generalized linear thermoelasticity. We use the non-linear Euler equations (inviscid flow) to demonstrate how our method can be applied after performing linearization of the flux Jacobian.

2. Spacetime discontinuous Galerkin methods

The spacetime discontinuous Galerkin methods developed in [4–9] are a family of discontinuous finite element methods for hyperbolic systems of equations. They utilize an advancing front mesh generation procedure that allows local 'patches' of elements to become decoupled from the global solution domain through causality. Other spacetime methods that use a more conventional 'time extrusion' or 'timeslab' approach can be found in [10–14]. Our formulations all share a control volume structure over a spacetime element, where volumetric changes are balanced by surface fluxes. We utilize the coordinate-free notation of differential forms, which is atypical in the computational mechanics literature but very useful on arbitrarily oriented manifolds in spacetime. The forms allow us to clearly identify and distinguish spatial and temporal fluxes, space-like and time-like manifolds, as well as circumventing the need to define a "natural" spacetime metric. The use of differential forms within the context of numerical methods has also been espoused by other authors, see e.g. [19]. Coordinate transformations are necessary on element faces, where we must solve the one-dimensional Riemann problem in the normal direction. As such, herein we provide details on the coordinate transformations used in our SDG implementations.

2.1. *Differential forms notation*

We use the notation of differential forms on spacetime manifolds to develop our SDG formulations. This approach supports a direct coordinate-free notation that can be used to express fluxes across spacetime interfaces with arbitrary orientation, such as element boundaries in unstructured spacetime meshes. This leads to concise representations of the governing equations that emphasize the notion of conservation on spacetime control volumes. In contrast to tensor notation, for example, the Stokes theorem expressed in forms notation does not require unit vectors 'normal' to spacetime d-manifolds. Such objects are not well defined, given the absence of an inner-product for spacetime vectors in classical mechanics. In this subsection, we present definitions and notations for differential forms with tensor coefficients on spacetime manifolds. See [21–24] for more complete expositions of differential forms and the exterior calculus on manifolds. Our formulation is specialized to flat spacetime manifolds for simplicity.

Consider a flat spacetime manifold $\mathcal{D} \subset \mathcal{M} := \mathbb{E}^d \times \mathbb{R}$ in which d is the spatial dimension of the manifold. We use the basis $\{\mathbf{e}_i, \mathbf{e}_t\}_{i=1}^d$, in which the spatial basis $\{\mathbf{e}_i\}$ spans $\mathbb{E}^d$ and $\mathbf{e}_t$ is the temporal basis vector, to represent *vectors* in the tangent space. The tangent bundle for our flat spacetime is uniform over $\mathcal{M}$, so we denote the tangent space at all points $P \in \mathcal{M}$ simply as $\mathcal{T}$, rather than the usual $\mathcal{T}_P$. The dual basis for *covectors* in the cotangent bundle $\mathcal{T}^*$ is denoted as $\{\boldsymbol{e}^i, \boldsymbol{e}^t\}_{i=1}^d$ and is determined by the relations $\boldsymbol{e}^i(\mathbf{e}_j) = \delta^i_j$, $\boldsymbol{e}^i(\mathbf{e}_t) = 0$, $\boldsymbol{e}^t(\mathbf{e}_i) = 0$ and $\boldsymbol{e}^t(\mathbf{e}_t) = 1$. Thus, the component representation of any vector $\mathbf{a} \in \mathcal{T}$ and any covector $\boldsymbol{b} \in \mathcal{T}^*$ are $\mathbf{a} = a^i\mathbf{e}_i + a^t\mathbf{e}_t$ and $\boldsymbol{b} = b_i\boldsymbol{e}^i + b_t\boldsymbol{e}^t$ in which, and from here on, summation from 1 to d is implied for indices repeated between subscripts and superscripts, excepting the reserved index t for which no summation is implied. We use bold italic type to denote forms and covectors and bold upright type to denote vectors and tensors.

Let $\mathcal{T}^r := \mathcal{T} \times \cdots \times \mathcal{T}$ (r times) be the *space of r-vectors*. The *space of r-covectors* (i.e., alternating, r-linear functions on $\mathcal{T}^r$) is denoted by $\Lambda^r\mathcal{T}^*$. The standard basis for r-covectors is denoted by $\{\boldsymbol{e}^\lambda\}$, in which $\lambda = i_1 \ldots i_r$ is a strictly increasing r-index. Any r-covector $\boldsymbol{\omega} \in \Lambda^r\mathcal{T}^*$ has a unique component representation with respect to the standard basis, $\boldsymbol{\omega} = \omega_\lambda\boldsymbol{e}^\lambda$, in which summation over strictly increasing r-indices is implied.

We use "$\wedge$" to denote the usual exterior product operator and $\mathbf{d}$ to denote the exterior derivative.

A *differential r-form on $\mathcal{D}$ (with scalar coefficients)* is an r-covector field on $\mathcal{D}$; we call these *r-forms* for short. The standard basis for 1-forms is $\{\mathrm{dx}^i, \mathrm{dt}\}_{i=1}^d$, where, for our flat manifold, the dx^i are 1-forms with uniform values $\boldsymbol{e}^i$, and dt is the 1-form with uniform value $\boldsymbol{e}^t$. Thus, any one form with scalar coefficients has the unique component representation with respect to the standard basis, $\boldsymbol{\omega} = \omega_i\mathrm{dx}^i + \omega_t\mathrm{dt}$, in which ω_i and ω_t are scalar fields on $\mathcal{D}$. Top forms in spacetime are $(d+1)$-forms, for which the standard basis is the singleton set $\{\boldsymbol{\Omega}\}$, where $\boldsymbol{\Omega} = \mathrm{dx}^1 \wedge \cdots \wedge \mathrm{dx}^d \wedge \mathrm{dt}$. Thus, a top-form $\boldsymbol{\alpha}$ with scalar coefficients is expressed as $\boldsymbol{\alpha} = \alpha\boldsymbol{\Omega}$ in which α is a scalar function on $\mathcal{D}$.

Let $\boldsymbol{\alpha}$ be an r-form with tensor coefficient $\mathbf{a}$ of order $s : s \in \mathbb{N}$, with a suitable inner product defined on the space of tensor coefficients. The *Hodge star operator* is defined by

$$\boldsymbol{\alpha} \wedge \star\boldsymbol{\alpha} = |\mathbf{a}|^2\boldsymbol{\Omega}, \tag{1}$$