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Harmonic mappings related to Janowski starlike functions

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ABSTRACT

The main purpose of the present paper is to give the extent idea which was introduced by Robinson (1947) [6]. One of the interesting application of this extent idea is an investigation of the class of harmonic mappings related to Janowski starlike functions.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$ be the open unit disc in the complex plane $\mathbb{C}$. A complex-valued harmonic function $f : \mathbb{D} \to \mathbb{C}$ has the representation

$$f = h + \overline{g} \tag{1.1}$$

where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ and have the following power series expansion,

$$h(z) = \sum_{n=0}^{\infty} a_n z^n,$$

$$g(z) = \sum_{n=0}^{\infty} b_n z^n, \quad z \in \mathbb{D},$$

where $a_n, b_n \in \mathbb{C}, n = 0, 1, 2, \ldots$. Choose (i.e., $b_0 = 0$) so the representation (1.1) is unique in $\mathbb{D}$ and is called the canonical representation of f [1].

For the univalent and sense-preserving harmonic functions f in $\mathbb{D}$, it is convenient to make further normalization (without loss of generality), $h(0) = 0$ (i.e., $a_0 = 0$) and $h'(0) = 1$ (i.e., $a_1 = 1$). The family of such functions f is denoted by S_H [2]. The family of all functions $f \in S_H$ with the additional property that $g'(0) = 1$ (i.e., $b_1 = 0$) is denoted by S_H^0 [2]. Observe that the classical family of univalent functions S consists of all functions $f \in S_H^0$ such that $g(z) = 0$. Thus it is clear that $S \subset S_H^0 \subset S_H$ [2].

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Let Ω be the family of functions $\phi(z)$ regular in the open unit disc $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for every $z \in \mathbb{D}$.

For arbitrary fixed numbers A, B, $-1 \leq B < A \leq 1$ denote by $P(A, B)$, the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ regular in $\mathbb{D}$ and such that $p(z)$ is in $P(A, B)$ if and only if

$$p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)}, \tag{1.2}$$

for some function $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$. This class was introduced by Janowski W. [3].

Lemma 1.1 ([4]). *Let $\phi(z)$ be regular in the open unit disc $\mathbb{D}$ with $\phi(0) = 0$ and $|\phi(z)| < 1$. If $|\phi(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_1, then we have $z_1.\phi'(z) = k\phi(z_1)$ for some $k \geq 1$.*

Theorem 1.2 ([3]). *If $h(z) \in S^*(A, B)$, then for $|z| = r$, $0 \leq r < 1$*

$$C(r, -A, -B) \leq |h(z)| \leq C(r, A, B)$$

$$C(r, A, B) = \begin{cases} r(1 + Br)^{\frac{A-B}{B}}, & B \neq 0; \\ re^{Ar}, & B = 0. \end{cases} \tag{1.3}$$

Theorem 1.3 ([5]). *Let $h(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ be an element of $S^*(A, B)$, then*

$$|a_n|^2 \leq \begin{cases} \displaystyle\prod_{m=0}^{k-2} \frac{|(A - B) + mB|^2}{(m + 1)^2}, & B \neq 0; \\ \displaystyle\prod_{m=0}^{k-2} \frac{|A|^2}{(m + 1)^2}, & B = 0. \end{cases} \tag{1.4}$$

2. Main results

Theorem 2.1. *Let $f = (h(z) + \overline{g(z)})$ be an element of $S_H^*(A, B)$. Then*

$$\frac{g(z)}{h(z)} \prec b_1 \frac{1 + Az}{1 + Bz}. \tag{2.1}$$

Proof. Since $f = h(z) + \overline{g(z)}$ is an element of $S_H^*(A, B)$, we have

$$\frac{g'(z)}{h'(z)} \prec b_1 \frac{1 + Az}{1 + Bz} \Rightarrow \frac{1}{b_1} \frac{g'(z)}{h'(z)} = \frac{1 + A\phi(z)}{1 + B\phi(z)} \Rightarrow \tag{2.2}$$

$$\left| \frac{1}{b_1} \frac{g'(z)}{h'(z)} - \frac{1 - ABr^2}{1 - B^2 r^2} \right| \leq \frac{(A - B)r}{1 - B^2 r^2} \Rightarrow$$

$$|b_1| \frac{1 - Ar}{1 - Br} \leq \left| \frac{g'(z)}{h'(z)} \right| \leq |b_1| \frac{1 + Ar}{1 + Br}.$$

Therefore the relations (2.2) shows that the values of $(\frac{g'(z)}{h'(z)})$ are in the disc

$$D_r(b_1) = \begin{cases} \left\{ \dfrac{g'(z)}{h'(z)} \middle| \left| \dfrac{g'(z)}{h'(z)} - \dfrac{b_1(1 - AB)r^2}{1 - B^2 r^2} \right| \leq \dfrac{|b_1|\,(1 - AB)r^2}{1 - B^2 r^2} \right\}, & B \neq 0 \\ \left\{ \dfrac{g'(z)}{h'(z)} \middle| \left| \dfrac{g'(z)}{h'(z)} - b_1 \right| \leq |b_1|\, Ar \right\}, & B = 0. \end{cases} \tag{2.3}$$

Now we define a function $\phi(z)$ by

$$\frac{g(z)}{h(z)} = b_1 \frac{1 + A\phi(z)}{1 + B\phi(z)} \tag{2.4}$$