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On the dispersion, stability and accuracy of a compact higher-order finite difference scheme for 3D acoustic wave equation

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ABSTRACT

In this paper, we propose a compact fourth-order finite difference scheme with low numerical dispersion to solve the 3D acoustic wave equation. Padé approximation has been used to obtain fourth-order accuracy in both temporal and spatial dimensions, while the alternating direction implicit (ADI) technique has been used to reduce the computational cost. Error analysis has been conducted to show the fourth-order accuracy, which has been confirmed by a numerical example. We have also shown that the proposed method is conditionally stable with a Courant–Friedrichs–Lewy (CFL) condition that is comparable to other existing finite difference schemes. Due to the higher-order accuracy, the new method is found effective in suppressing numerical dispersion.

1. Introduction

Finite difference (FD) schemes have been widely used by Geophysicists, Mathematicians and engineers to find the numerical solutions of seismic wave equations, as the analytical solutions to such mathematical models are not available in general. Among them, higher-order methods have attracted the interests of many researchers working on seismic modeling (see [1–10] and references therein).

Recently, a great deal of effort has been devoted to develop higher-order FD schemes for the acoustic equations mentioned above, and many accurate and efficient methods haven been reported. Levander [11] addressed the cost-effectiveness of solving real problems using higher-order spatial derivatives to allow a more coarse spatial sample rate. In [5], the authors used a plane wave theory and the Taylor series expansion to develop a low dispersive time–space domain FD scheme with error of $O(\tau^2 + h^{2M})$ for 1-D, 2-D and 3-D acoustic wave equations, where τ and h represent the time step and spatial grid size, respectively. It was then shown that, along certain fixed directions the error can be improved to $O(\tau^{2M} + h^{2M})$. In [2], Cohen and Poly extended the works of Dablain [12], Shubin and Bell [13] and Bayliss et al. [14] and developed a fourth-order accurate explicit scheme with error $O(\tau^4 + h^4)$ to solve the heterogeneous acoustic wave equation. Moreover, it has been reported that highly accurate numerical methods are very effective in suppressing the annoying numerical dispersion [15–17].

These methods are accurate but are non-compact, which give rise to two issues: efficiency and difficulty in boundary condition treatment. To resolve these issues, a variety of compact higher-order FD schemes to approximate spatial derivatives have been developed. In [18], the authors developed a family of fourth-order three-point combined difference schemes to approximate the first- and second-order spatial derivatives. In [19], the authors introduced a family of three-level implicit FD schemes which incorporate the locally one-dimensional method. For more recent compact higher-order difference methods, the readers are referred to [4,20,21].

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It is well-known that for multi-dimensional problems, a block tridiagonal system needs to be solved at each time step. To efficiently solve such a large linear system, some operator splitting techniques are required to break the multi-dimensional problem into a sequence of one-dimensional problems. One such operator splitting method is the alternating direction implicit (ADI) method, which was originally introduced by Peaceman and Rachford [22] to solve parabolic and elliptic equations. It has witnessed a lot of development over the years for hyperbolic equations as well [23,24]. For example, Fairweather and Mitchell [25] developed a fourth-order compact ADI scheme for solving the wave equation. More related work can be found in [26,27].

In this paper, we extend the Padé approximation based higher-order compact FD scheme in [28] to a 3D acoustic wave equation and obtain a compact efficient method, which is fourth-order accurate in both time and space. We first give a brief introduction of the standard second-order central FD scheme, compact higher-order FD scheme for spatial derivatives and some other related higher-order method, then derive the new compact fourth-order ADI FD scheme in Section 2, which is followed by the numerical dispersion analysis and stability analysis in Sections 3 and 4, respectively. We then demonstrate the accuracy and efficiency of the new scheme by applying it to solve several numerical examples in Section 5. Finally, conclusions and possible future extensions are discussed in Section 6.

2. The new compact higher-order ADI scheme

Consider the 3D acoustic wave equation given by

$$u_{tt} = v^2(u_{xx} + u_{yy} + u_{zz}), \quad (x, y, z, t) \in \Omega \times [0, T], \tag{1}$$

$$u(x, y, z, 0) = f_1(x, y, z), \quad (x, y, z) \in \Omega, \tag{2}$$

$$u_t(x, y, z, 0) = f_2(x, y, z), \quad (x, y, z) \in \Omega, \tag{3}$$

$$u(x, y, z, t) = g(x, y, z, t), \quad (x, y, z, t) \in \partial\Omega \times [0, T], \tag{4}$$

where v is the wave velocity and $\Omega \subset R^3$ is a finite domain.

For the sake of simplicity, we assume that Ω is a cubic domain that will be discretized into an $N_x \times N_y \times N_z$ grid with spatial grid sizes h_x, h_y and h_z. Let τ denote the time step, $u^n_{i,j,k}$ denote the numerical solution at the grid point (x_i, y_j, z_k) and time level $n\tau$. In the standard central FD scheme, all second derivatives in Eq. (1) are approximated by the following second-order central difference formulas:

$$u_{tt}(x_i, y_j, z_k, t_n) \approx \delta_t^2 u^n_{i,j,k}/\tau^2 = (u^{n-1}_{i,j,k} - 2u^n_{i,j,k} + u^{n+1}_{i,j,k})/\tau^2, \tag{5}$$

$$u_{xx}(x_i, y_j, z_k, t_n) \approx \delta_x^2 u^n_{i,j,k}/h_x^2 = (u^n_{i-1,j,k} - 2u^n_{i,j,k} + u^n_{i+1,j,k})/h_x^2, \tag{6}$$

$$u_{yy}(x_i, y_j, z_k, t_n) \approx \delta_y^2 u^n_{i,j,k}/h_y^2 = (u^n_{i,j-1,k} - 2u^n_{i,j,k} + u^n_{i,j+1,k})/h_y^2, \tag{7}$$

$$u_{zz}(x_i, y_j, z_k, t_n) \approx \delta_z^2 u^n_{i,j,k}/h_z^2 = (u^n_{i,j,k-1} - 2u^n_{i,j,k} + u^n_{i,j,k+1})/h_z^2. \tag{8}$$

To obtain a higher-order scheme, one needs to approximate these derivatives with higher-order accuracy. The conventional higher-order FD method was derived by approximating the spatial derivatives with higher-order accuracy using more than three points in one direction, which results in a large stencil. For instance, if $2M + 1$ points are used to approximate u_{xx}, one obtains the following formula

$$u_{xx}(x_i, y_j, z_k, \tau_n) \approx \frac{1}{h_x^2}\left[a_0 u^n_{i,j,k} + \sum_{m=1}^{M} a_m(u^n_{i-m,j,k} + u^n_{i+m,j,k})\right], \tag{9}$$

which can be as accurate as the $(2M)$th-order in space. The conventional high-order FD method is accurate in space but suffers severe numerical dispersion, as shown later. Another issue is that it requires large computer memory due to the large stencil.

To improve the accuracy in time, a class of time-domain high-order FD methods haven been derived by Liu and Sen [5,6]. The idea of the time-domain high-order FD method is to determine coefficients using both time and space domain. As a result, the coefficient a_m will be a function of $r(=\frac{v\tau}{h})$. It was noted that in the 1D case, the time-domain high-order FD method can be as accurate as the $(2M)$th-order in both time and space, provided some conditions on r are satisfied, while for the multidimensional case, the $(2M)$th-order is also possible along some particular directions.

To develop the higher-order compact ADI FD scheme, we first apply the Padé approximation to the second-order central FD operators defined in Eqs. (5)–(8), so the second derivatives u_{xx}, u_{yy}, u_{zz} and u_{tt} are approximated with fourth order accuracy in space and time, respectively:

$$\frac{\delta_x^2}{h_x^2\left(1 + \frac{1}{12}\delta_x^2\right)} u^n_{i,j,k} = u_{xx}(x_i, y_j, z_k, t_n) + O(h_x^4), \tag{10}$$

$$\frac{\delta_y^2}{h_y^2\left(1 + \frac{1}{12}\delta_y^2\right)} u^n_{i,j,k} = u_{yy}(x_i, y_j, z_k, t_n) + O(h_y^4), \tag{11}$$