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A note on the adaptive conservative/dissipative discretization for evolutionary partial differential equations

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ABSTRACT

An adaptive conservative or dissipative numerical method for nonlinear partial differential equations is established. The method not only inherits the conservation or dissipation property of the equation but also uses suitable non-uniform grids at each time step. Our numerical experiments indicate that the method is useful especially for localized solutions such as solitary wave solutions.

1. Introduction

In this note, we show that by a simple idea we can establish an adaptive conservative or dissipative numerical method for partial differential equations (PDEs) with the independent variables $(x, t) \in \mathbb{R}^d \times \mathbb{R}$, of the form

$$\frac{\partial u}{\partial t} = \mathcal{D}\frac{\delta G}{\delta u}, \tag{1}$$

where $u = u(x, t) \in \mathbb{R}$, $\mathcal{D}$ is a skew-symmetric or negative semidefinite differential operator, and $\delta G/\delta u$ denotes the variational derivative of $G(u, u_x)$. In the numerical analysis of differential equations, "structure-preserving" methods have been attracting much attention. They are methods preserving geometric properties of a differential equation (for example, see [1] for ODEs and [2,3] for PDEs). In this note we restrict our attention to PDEs of the form (1). If $\mathcal{D}$ is skew-symmetric, (1) has a conservation property

$$\frac{\mathrm{d}}{\mathrm{d}t}\int G(u, u_x)\mathrm{d}x = 0,$$

under appropriate boundary conditions. A typical example of this class is the KdV equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(3u^2 + \frac{\partial^2 u}{\partial x^2}\right), \quad 0 < x < L,\ t > 0, \tag{2}$$

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where $G(u, u_x) = u^3 - u_x^2/2$ and L is a real number. If $\mathcal{D}$ is negative semidefinite, (1) has a dissipation property

$$\frac{\mathrm{d}}{\mathrm{d}t}\int G(u, u_x)\mathrm{d}x \le 0,$$

again under appropriate boundary conditions.[1] A typical example of this class is the Cahn–Hilliard equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}\left(pu + ru^3 + q\frac{\partial^2 u}{\partial x^2}\right), \quad 0 < x < L,\ t > 0, \tag{3}$$

where $G(u, u_x) = pu^2/2 + ru^4/4 - qu_x^2/2$, and p, q, r are real parameters.

In the last two decades, much effort has been devoted in order to construct several frameworks which derive conservative/dissipative schemes. For example, Furihata proposed the discrete variational derivative method (DVDM) [4] (see also Furihata–Matsuo [5], Celledoni et al. [6]) in finite difference context. It has then been applied to some fundamental PDEs to prove that the method is in fact effective. We note that the concept of the temporal discretization is essentially the same as the discrete gradient method known in the ODE context (for the discrete gradient method, see [7–9] for example).

However, there remained several issues to be settled before the method could be truly useful for large, practical applications. The first issue was the adaptation to non-uniform grids. The original DVDM was constructed only on uniform grids since it required the summation-by-parts formula regarding difference operators. Obviously such formulas are not easily constructed on non-uniform grids. Fortunately, this issue has been successfully settled by some recent studies. Yaguchi–Matsuo–Sugihara found their way by using either the mapping method [10] or discrete differential forms [11]. Matsuo [12] gave another solution by extending the DVDM to Galerkin (finite element) context.

Another difficulty in the original DVDM was that it assumed *static* grids, and it was not clear at all if it could be incorporated with a *dynamic* grid technique. Such a technique is required in practical problems where a localized point (or area) moves as time passes (consider, for example, a moving solitary wave), in order to increase the overall efficiency. Unfortunately, however, it seems that no study has ever succeeded in such a challenge, not only in the context of the DVDM, but also in the more general context of the structure-preserving methods for PDEs, except in very specific studies such as [13]. The reason for this is that such structure-preserving methods usually employ a very sophisticated time stepping for the desired structure-preservation, which generally seems to contradict the concept of grid adaptation.

Motivated by this background, in this note we shall show that by a simple idea we can establish an adaptive conservative/dissipative method. This is done by combining the following two main techniques: the conservative/dissipative method on *static* non-uniform grids mentioned above, and the grid adaptation technique frequently used in the context of the wavelet based numerical methods [14–17]. Here we would like to emphasize that a simple combination of them would destroy the desired conservation/dissipation properties. The key is to introduce an additional optimization step, by which the destruction can be avoided. As far as the authors know, this is the first study where a systematic grid adaptation is realized in the context of energy conservative/dissipative methods for PDEs.

This note is organized as follows. In Section 2, the standard conservative/dissipative method on non-uniform grids is reviewed. As an example, we employ the Galerkin approach [12]. In Section 3, the standard dynamic grid adaptation technique is reviewed to show how to obtain appropriate grids at each time step. In Section 4, the adaptive conservative/dissipative algorithm and numerical experiments are shown. Conclusions are discussed in Section 5.

Throughout this note, numerical solutions are denoted by $u^{(n)} \simeq u(n\Delta t, \cdot)$ where Δt is the time step size, and the inner product is defined by $(f, g) = \int_0^L fg\mathrm{d}x$. Although the ideas in the present paper should carry to two- or three-dimensional cases, we restrict ourselves to one-dimensional problems for the clarity of description.

2. Energy conservative/dissipative method on static non-uniform grids

In this section, we review the energy conservative/dissipative Galerkin method [12] for the KdV and Cahn–Hilliard equations.

Suppose that the interval $[0, L]$ is partitioned appropriately (not necessarily uniformly), and let $S_h \subset H^1(0, L)$ (H^1 denotes the first order Sobolev space) be, for example, the piecewise linear function space over the grid. For the KdV equation (2), let us use the space $X = \{v \mid v \in S_h,\ v(0) = v(L)\}$ in order to consider the periodic boundary conditions. The KdV equation can be written as the variational (Hamiltonian) form

$$u_t = \partial_x \frac{\delta G}{\delta u}, \quad G(u, u_x) = u^3 - \frac{u_x^2}{2},$$

or equivalently, the following system:

$$u_t = (p_1)_x, \quad p_1 = \frac{\delta G}{\delta u}.$$

[1] Hereafter $G(u, u_x)$ is often abbreviated as $G(u)$ when no confusion can occur.