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A family of multivariate multiquadric quasi-interpolation operators with higher degree polynomial reproduction

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HIGHLIGHTS

- A family of multivariate multiquadric quasi-interpolants is proposed.
- There is no demand for derivatives of approximated function in quasi-interpolants.
- The quasi-interpolants satisfy any degree polynomial reproduction property.
- The quasi-interpolants can reach up to a higher approximation order.

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ABSTRACT

In this paper, by using multivariate divided difference (Rabut, 2001) to approximate the partial derivative and the idea of the superposition (Waldron, 2009), we modify a multiquadric quasi-interpolation operator (Ling, 2004) based on a dimension-splitting technique with the property of linear reproducing to gridded data on multi-dimensional spaces, such that a family of proposed multivariate multiquadric quasi-interpolation operators Φ_{r+1} has the property of $r+1$ ($r \in \mathbb{Z}, r \geqslant 0$) degree polynomial reproducing and converges up to a rate of $r+2$. In addition, the proposed quasi-interpolation operator only demands information of location points rather than the derivatives of the function approximated. Moreover, we give the approximation error of our quasi-interpolation operator. Finally, some numerical experiments are shown to confirm the approximation capacity of our quasi-interpolation operator.

1. Introduction

The approximation of the multivariate function from scattered data is an important theme in numerical mathematics. Although the classical method to attack the problem is interpolation, the interpolation matrix quickly becomes ill-conditioned as the number of points increases. There are different ways to overcome this ill-conditioned problem. Compared with interpolation, the quasi-interpolation method does not require solution of any linear system, and can reach up to the expected convergence rate. In this paper, we will focus on the quasi-interpolation method. For a set of functional values $\{f(q_j)\}_{j=1}^{n}$ taken on a set of nodes $\Xi = \{\boldsymbol{q}_0, \ldots, \boldsymbol{q}_n\} \subseteq \mathbb{R}^d$ and a set of given quasi-interpolation basis functions $\Psi(\boldsymbol{q})$, the

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quasi-interpolation $(\mathcal{Q}f)(\boldsymbol{q})$ of a d-variate function f takes the standard form via linear combination

$$(\mathcal{Q}f)(\boldsymbol{q}) = \sum_{j=0}^{n} f(\boldsymbol{q}_j)\Psi(\boldsymbol{q}), \quad \boldsymbol{q} \in \mathbb{R}^d. \tag{1}$$

The approximation properties of the quasi-interpolation operator in the case that $\boldsymbol{q}_j$ are nodes of a uniform grid can be well-understood. For example, for a uniform grid with spacing h, the following quasi-interpolation operator

$$(\mathcal{Q}f)(\boldsymbol{q}) = \sum_{j\in\mathbb{Z}^d} f(jh)\Psi\left(\frac{\boldsymbol{q}}{h} - j\right) \tag{2}$$

proposed by Rabut [1] can be studied via the theory of principal shift-invariant spaces (see, e.g., [2]), which has been developed by de Boor et al. [3]. Here, the quasi-interpolation basis function Ψ is supposed to be a compactly supported or rapidly decaying function. In [4], Bozzini, Dyn, and Rossini presented a procedure in spaces of m-harmonic splines in $\mathbb{R}^d$ which starts from a simple generator Ψ_0 and recursively defines generators $\Psi_1, \Psi_2, \ldots, \Psi_{m-1}$ with corresponding quasi-interpolation operators defined as (2) reproducing polynomials of degrees $3, 5, \ldots, 2m-1$ respectively. In [5,6], the Strang–Fix condition is considered a necessary and sufficient condition for the polynomial reproduction, convergence and approximation order of the quasi-interpolation. Based on the Strang–Fix condition for Ψ, the scattered data quasi-interpolation by functions, which reproduces polynomials, has been proposed by Buhmann et al. [7], Dyn and Ron [8], Wu and Liu [9], and Wu and Xiong [10]. We will discuss the multivariate multiquadric quasi-interpolation operator which can reproduce higher degree polynomials in this paper.

With the multiquadric function

$$\varphi_i(x) = \left[(x - x_i)^2 + c^2\right]^{1/2}, \quad x, x_i \in \mathbb{R}, c > 0 \tag{3}$$

proposed by Hardy [11], Beast and Powell [12] first constructed a univariate quasi-interpolation operator by shifts of the multiquadric basis function of first degree to finite scattered data, which reproduces linear polynomials. However, the operator requires the derivatives of the function approximated at endpoints, which is not convenient for practical purposes. Thus, Wu and Schaback [13] constructed another univariate multiquadric quasi-interpolation operator with modifications at endpoints without derivatives.

Ling [14] extended the univariate quasi-interpolation operator (see, e.g., [13]) to multi-dimension by using the dimension-splitting univariate multiquadric basis function approach, which reproduces linear polynomials. Based on [14], Feng and Zhou [15] proposed a multivariate dimension-splitting multiquadric quasi-interpolation operator satisfying quadratic polynomial reproduction property, and proved that it converged with a rate of $\mathcal{O}(h^3)$ as $c = \mathcal{O}(h)$, $h = \max\{h_1, h_2\}$, $2h_1 = \max_{0\leqslant i\leqslant n-1}\{|x_{i+1} - x_i|\}$, $2h_2 = \max_{0\leqslant j\leqslant m-1}\{|y_{j+1} - y_j|\}$. In [16], for any given quasi-interpolation operator which reproduces all polynomials of degree $\leqslant m$, Waldron proposed a quasi-interpolation operator which reproduces all polynomials of degree $\leqslant m + r$, where r is a non-negative integer. However, it involves the derivatives of the function approximated at each knot.

The aim of our paper is to present a family of multivariate multiquadric quasi-interpolation operators with higher accuracy for gridded data, which does not require the partial derivatives of the function approximated at each knot. By virtue of the constructive idea in [16] and the multivariate divided difference formula in [17], we extend the operator [14] and give a family of multivariate multiquadric quasi-interpolation operators based on the dimension-splitting technique for gridded data. We show that the new operator could reproduce any degree polynomials and have the relevant order approximation rate under a certain assumption. Our error estimate indicates that the convergence rate depends heavily on the shape parameter c. Thus, our operator could provide the desired smoothness and precision by choosing a suitable value of the shape parameter c.

The remainder of this paper is organized as follows. In Section 2, we introduce some necessary definitions. In Section 3, we propose a family of multivariate multiquadric quasi-interpolation operators satisfying any degree polynomial reproduction property. In Section 4, we obtain the error estimate. In Section 5, we give some numerical experiments, and verify the relationship between the approximation order and the degree of polynomial reproducing. In Section 6, we give the conclusions, and put forward the future work.

2. Preliminaries

Definition 1 (*[15]*)**.** Let $k \in \mathbb{N}$, $c > 0$. The multiquadric functions of $2k - 1$ degree ($2k$ order) is defined by

$$\varphi(\boldsymbol{q}; 2k) = \left(\|\boldsymbol{q}\|^2 + c^2\right)^{(2k-1)/2}, \quad \boldsymbol{q} = (x_1, \ldots, x_d) \in \mathbb{R}^d, \tag{4}$$

where c is called the shape parameter, $\mathbb{R}^d$ is the d-dimensional Euclidean space, $\|\cdot\|$ is the Euclidean norm, and $\varphi_{j,2k}(\boldsymbol{q})$ is a shift of $\varphi(\boldsymbol{q}; 2k)$ centered at $\boldsymbol{q}_j$:

$$\varphi_{j,2k}(\boldsymbol{q}) = \varphi(\boldsymbol{q} - \boldsymbol{q}_j; 2k), \quad \boldsymbol{q}_j = (x_{j1}, \ldots, x_{jd}) \in \mathbb{R}^d. \tag{5}$$