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Planar quintic G^2 Hermite interpolation with minimum strain energy

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ABSTRACT

In this paper, we study planar quintic G^2 Hermite interpolation with minimum strain energy. To match arbitrary G^2 Hermite data, a quintic curve is expressed in terms of four free parameters that encode the local reparameterization at the endpoints and are available for further optimization. We express the approximate strain energy as a quartic function in four parameters, whose minimum can be found by solving an optimization problem of two parameters relating to the magnitudes of endpoint tangent vectors. A feasible region is used while searching the optimal values of these two parameters such that the interpolating curve can preserve tangent directions and avoid singularities at the endpoints. We then solve this constrained minimization problem via the proximal gradient method. Several comparative examples are provided to demonstrate the effectiveness of the proposed method and applications to shape design are also shown.

1. Introduction

The construction of fair curves and surfaces is a fundamental problem in the field of geometric modeling and related applications [1,2]. To satisfy functional or aesthetic criteria, designed objects often have to exactly match prescribed data, such as a series of points and derivatives. In industrial design, fair curves and surfaces are the most preferred representation to meet the requirements of design and modeling. A curve or surface is represented using more than enough degrees of freedom to satisfy demanded geometric constraints, with the remaining freedom used to achieve a fair shape by minimizing an energy function representing the fairness, see e.g. [3–7].

The standard Hermite interpolation is simple to construct and compute, but it does not always provide satisfactory results. As pointed out in [3], C^1 Hermite interpolation may produce undesired shapes if the magnitudes of tangent vectors are unsuitable. Geometric Hermite interpolation is a natural generalization which deals with interpolation of geometric quantities such as points, tangent directions, curvatures, and so on. de Boor et al. [8] proposed a cubic G^2 interpolation scheme for planar curves, but the solutions that are computed numerically exist only when admissibility conditions on the data are satisfied. This method is generalized to the quartic case [9]: if the endpoint tangent directions have an intersection, a family of quartic curves with two free parameters is constructed.

A lot of work on G^2 Hermite interpolation using spiral segments has been proposed. Spiral segments are a kind of geometric curves with monotone curvature and have been widely considered as fair curves in path design and computer graphics [10]. Since non-uniform rational *B*-splines (NURBS) are the industry standard for the representation of geometry in CAD/CAM, many researchers have proposed to use polynomial or rational curves of low degrees for matching G^2 Hermite data. For example, there are many approaches based on cubics [11], Pythagorean hodograph (PH) quintics [12] and rational

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cubics [13]. However, these methods have to meet certain conditions under which the interpolating curves are guaranteed to be spiral segments — that is, only admissible G^2 Hermite data can be interpolated by polynomial spiral segments. It is rather annoying when encountering the case where the required conditions are not satisfied. Recently, Deng and Ma [14] presented a biarc-based subdivision scheme that can produce planar spirals without this limitation, but the resulting subdivision curves are non-polynomial.

In this paper, we consider the problem of G^2 Hermite interpolation by quintic curves. Yong and Cheng [3] introduced cubic optimized geometric Hermite (OGH) curves that interpolate G^1 data. Their main idea is to optimize the magnitudes of endpoint tangent vectors so that the strain energy of a cubic curve is minimized. Jaklič and Žagar [15] constructed cubic G^1 interpolatory splines by taking tangent directions as unknowns, thereby relaxing conditions on admissible regions for tangent directions. In [16], optimized geometric Hermite curves are obtained by minimizing the curvature variation. However, these methods have two limitations. First, the preservation of tangent directions, which is deemed highly desirable in shape design and modeling applications, cannot be guaranteed if tangent angle constraints are not satisfied. Second, singular cases arise when the magnitudes of tangent vectors approach zero. For example, when the two tangent angles are very close to $\pi/2$, the resulting curve is nearly a straight line segment. To overcome these problems mentioned above, a possible way is to use polynomial curves of higher degree. It is clear that the quintics are the lowest degree polynomial curves that can match arbitrary G^2 Hermite data without any constraint and have enough flexibility for shape design. A quintic curve has four more degrees of freedom than a cubic curve, but two degrees of freedom are used to match given curvatures in G^2 Hermite interpolation. Furthermore, we will show that, for arbitrary G^2 Hermite data, a quintic interpolating curve can always be constructed in terms of four free parameters that encode the local reparameterization at the endpoints [17]. In contrast to the previous methods [8,9], our method has no constraint on the input data and thus can be applied widely.

Our goal is to determine the four free parameters by minimizing the strain energy. It is an energy minimization based method to obtain fair curves, and generalizes cubic G^1 Hermite interpolation in [3,15,16] to quintic G^2 Hermite interpolation. We observe that two parameters are quadratic functions of the other two parameters when the minimum is reached, and thus express the approximate strain energy as a quartic function of two unknowns. We say that it is an *acceptable* solution if the endpoint tangent vectors of the resulting curve have the given directions and the magnitudes lying in a user-specified feasible region. Therefore, it is equivalent to solving a constrained minimization problem subject to the trust-region bound, which can be efficiently solved by using the *proximal gradient* method (see [18,19] for details).

The rest of this paper is organized as follows. In Section 2 we represent a quintic curve with four parameters to match G^2 Hermite data. In Section 3 we present the algorithm based on the proximal gradient method. In Section 4 we show several examples and comparisons with the previous methods. Finally, we conclude the paper in Section 5.

2. Quintic G^2 Hermite interpolation

A planar quintic Bézier curve is defined by

$$\mathbf{b}(t) = \sum_{i=0}^{5} \mathbf{b}_i B_i^5(t), \quad t \in [0, 1], \tag{1}$$

where $\mathbf{b}_i \in \mathbb{R}^2$ are control points and $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$ are Bernstein polynomials of degree n. The signed curvature of $\mathbf{b}(t)$ is

$$\kappa(t) = \frac{\det(\mathbf{b}'(t), \mathbf{b}''(t))}{\|\mathbf{b}'(t)\|^3}$$

if $\mathbf{b}(t)$ is regular, i.e., $\|\mathbf{b}'(t)\| \neq 0$. The curvature is positive if the center of the osculating circle is on the left when a curve is traversed in the direction of increasing parameter; otherwise, it is negative.

Planar G^2 Hermite data consist of two points $\mathbf{P}_0, \mathbf{P}_1$, with the associated unit tangent vectors $\mathbf{T}_0, \mathbf{T}_1$ and curvatures κ_0, κ_1. Denote by φ_0 the angle from $\mathbf{T}_0$ to $\mathbf{P}_0\mathbf{P}_1$ and by φ_1 the angle from $\mathbf{P}_0\mathbf{P}_1$ to $\mathbf{T}_1$, where counterclockwise angles are positive and clockwise angles are negative. Let $\mathbf{N}_0$ and $\mathbf{N}_1$ be the unit normals formed by rotating $\mathbf{T}_0$ and $\mathbf{T}_1$ counterclockwise through $\pi/2$, respectively. If the curvature at an endpoint is positive (or negative), the curvature vector and normal vector at this endpoint have the same (or opposite) directions, see Fig. 1 for an illustration. In practical applications, there are two important transition curves: if $\varphi_0 > 0, \varphi_1 > 0, \kappa_0 > 0, \kappa_1 > 0$ or $\varphi_0 < 0, \varphi_1 < 0, \kappa_0 < 0, \kappa_1 < 0$, then a C-shaped transition curve is sought; if $\varphi_0 > 0, \varphi_1 < 0, \kappa_0 > 0, \kappa_1 < 0$ or $\varphi_0 < 0, \varphi_1 > 0, \kappa_0 < 0, \kappa_1 > 0$, then an S-shaped transition curve is sought.

To match the positions and tangent directions at the endpoints, it yields

$$\mathbf{b}_0 = \mathbf{P}_0, \qquad \mathbf{b}_5 = \mathbf{P}_1 \tag{2}$$

and

$$\mathbf{b}_1 = \mathbf{P}_0 + \frac{\alpha_0}{5}\mathbf{T}_0, \qquad \mathbf{b}_4 = \mathbf{P}_1 - \frac{\alpha_1}{5}\mathbf{T}_1 \tag{3}$$

where $\alpha_i > 0$ are scalar parameters.