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A reduced fast component-by-component construction of lattice points for integration in weighted spaces with fast decreasing weights

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ABSTRACT

Lattice rules and polynomial lattice rules are quadrature rules for approximating integrals over the s-dimensional unit cube. Since no explicit constructions of such quadrature methods are known for dimensions $s > 2$, one usually has to resort to computer search algorithms. The fast component-by-component approach is a useful algorithm for finding suitable quadrature rules.

We present a modification of the fast component-by-component algorithm which yields savings of the construction cost for (polynomial) lattice rules in weighted function spaces. The idea is to reduce the size of the search space for coordinates which are associated with small weights and are therefore of less importance to the overall error compared to coordinates associated with large weights. We analyze tractability conditions of the resulting QMC rules. Numerical results demonstrate the effectiveness of our method.

1. Introduction

In this paper we study the construction of quasi-Monte Carlo (QMC) rules

$$\frac{1}{N}\sum_{n=0}^{N-1} f(\boldsymbol{x}_n) \approx \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \tag{1}$$

which are used for the approximation of s-dimensional integrals over the unit cube $[0, 1]^s$. We consider two types of quadrature point sets $\mathcal{P} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}\}$, namely, lattice point sets [1–3] and polynomial lattice point sets [2,4,5]. For a natural number $N \in \mathbb{N}$ and a vector $\boldsymbol{z} \in \{1, 2, \ldots, N-1\}^s$, a lattice point set is of the form

$$\left\{\frac{k}{N}\boldsymbol{z}\right\} \quad \text{for } k = 0, 1, \ldots, N-1.$$

Here, for real numbers $x \geq 0$ we write $\{x\} = x - \lfloor x \rfloor$ for the fractional part of x. For vectors $\boldsymbol{x}$ we apply $\{\cdot\}$ component-wise. Polynomial lattice point sets are similar, one only replaces the arithmetic over the real numbers by arithmetic of polynomials over finite fields. More details about polynomial lattice point sets are given in Section 5.

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In order to analyze the quality of a given point set $\mathcal{P}$ with respect to its performance in a QMC rule (1), one usually considers the worst-case integration error in the unit ball of a Banach space $(\mathcal{H}, \|\cdot\|)$ given by

$$e_{N,s}(\mathcal{H},\mathcal{P}) = \sup_{\substack{f\in\mathcal{H}\\ \|f\|\leq 1}} \left| \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{|\mathcal{P}|}\sum_{\boldsymbol{y}\in\mathcal{P}} f(\boldsymbol{y}) \right|.$$

Here we restrict ourselves to certain weighted reproducing kernel Hilbert spaces $\mathcal{H}$. For background on reproducing kernel Hilbert spaces see [6] and for reproducing kernel Hilbert spaces in the context of numerical integration see [7,8]. In case of lattice rules, we consider the so-called weighted Korobov space (Section 2) and for polynomial lattice rules we consider weighted Walsh spaces (Section 5).

The paper [8] introduced weighted reproducing kernel Hilbert spaces. The weights are a set of non-negative real numbers $(\gamma_{\mathfrak{u}})_{\mathfrak{u}\subseteq[s]}$, where $[s] = \{1, 2, \ldots, s\}$, which model the importance of the projection of the integrand f onto the variables x_j for $j \in \mathfrak{u}$. A small weight $\gamma_{\mathfrak{u}}$ means that the projection onto the variables in $\mathfrak{u}$ contributes little to the integration problem. A simple choice of weights are so-called product weights $(\gamma_j)_{j\in\mathbb{N}}$, where $\gamma_{\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_j$. In this case, the weight γ_j is associated with the variable x_j.

We introduce the concept of tractability [9]. Let $e(N, s)$ be the Nth minimal QMC worst-case error

$$e(N, s) = \inf_{\mathcal{P}} e_{N,s}(\mathcal{H}, \mathcal{P}),$$

where the infimum is extended over all N-element point sets $\mathcal{P}$ in $[0, 1]^s$. We also define the initial error $e(0, s)$ as the integration error when approximating the integral by 0, that is,

$$e(0,s) = \sup_{\substack{f\in\mathcal{H}\\ \|f\|\leq 1}} \left| \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \right|.$$

This is used as a reference value.

We are interested in the dependence of the Nth minimal worst-case error on the dimension s. We consider the QMC information complexity, which is defined by

$$N_{\min}(\varepsilon, s) = \min\{N \in \mathbb{N} : e(N, s) \leq \varepsilon e(0, s)\}.$$

This means that $N_{\min}(\varepsilon, s)$ is the minimal number of points which are required to reduce the initial error by a factor of ε.

We can now define the following notions of tractability. We say that the integration problem in $\mathcal{H}$ is

1. weakly QMC tractable, if
$$\lim_{s+\varepsilon^{-1}\to\infty} \frac{\log N_{\min}(\varepsilon, s)}{s + \varepsilon^{-1}} = 0;$$
2. polynomially QMC-tractable, if there exist non-negative numbers c, p and q such that
$$N_{\min}(\varepsilon, s) \leq c s^q \varepsilon^{-p}, \tag{2}$$
and we call the infima over all q and p such that (2) holds the s- and ε-exponent of polynomial tractability, respectively;
3. strongly polynomially QMC-tractable, if (2) holds with $q = 0$. In this case, we call the infimum over all p such that (2) holds the ε-exponent of strong polynomial tractability.

It is known that, in order to achieve strong polynomial tractability of the integration problem in the weighted Korobov space with product weights $(\gamma_j)_{j\in\mathbb{N}}$, it is necessary and sufficient (see [10]) to have

$$\sum_{j=1}^{\infty} \gamma_j < \infty.$$

It is also well known, see [11], that if $\sum_{j=1}^{\infty} \gamma_j^{1/\tau} < \infty$ for some $1 \leq \tau < \alpha$, then one can set the ε-exponent to $2/\tau$. Here $\alpha > 1$ is the smoothness parameter of the elements of the Korobov space (we shall give more details about this in Section 2). In [11] for N prime and in [12] for arbitrary N, it was shown that suitable lattice rules can be constructed component-by-component. The construction cost of this algorithm was reduced to $O(sN \log N)$ operations by [13,14]. Assume now that

$$\sum_{j=1}^{\infty} \gamma_j^{1/\tau} < \infty \tag{3}$$

for some $\tau > \alpha$. Then no further advantage is obtained over the results from [11–14], since one still gets strong polynomial tractability with the optimal ε-exponent and the construction cost of the lattice rule is independent of the choice of weights. But since (3) implies that the importance of coordinates with bigger index is much smaller than that of earlier ones, it is unreasonable to spend the same amount of work to search for the corresponding component of the generating vector. The aim of this paper is to take advantage of a situation where (3) holds for $\tau > \alpha$, by showing that in this case one can reduce the construction cost of the lattice rule by making the search space for later components smaller than for earlier ones, while still achieving strong polynomial tractability with the optimal ε-exponent.