Journal of Computational and Applied Mathematics 276 (2015) 16–29

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Numerical stationary distribution and its convergence for nonlinear stochastic differential equations

Wei Liu [a,*], Xuerong Mao [b,c]

[a] Loughborough University, Department of Mathematical Sciences, Loughborough, Leicestershire, LE11 3TU, UK
[b] University of Strathclyde, Department of Mathematics and Statistics, Glasgow, G1 1XH, UK
[c] School of Economics and Management, Fuzhou University, China

ARTICLE INFO

Article history:
Received 18 February 2014
Received in revised form 27 July 2014

Keywords:
The backward Euler–Maruyama method
Nonlinear SDEs
Numerical stationary distribution
Weak convergence

ABSTRACT

To avoid finding the stationary distributions of stochastic differential equations by solving the nontrivial Kolmogorov–Fokker–Planck equations, the numerical stationary distributions are used as the approximations instead. This paper is devoted to approximate the stationary distribution of the underlying equation by the Backward Euler–Maruyama method. Currently existing results (Mao et al., 2005; Yuan et al., 2005; Yuan et al., 2004) are extended in this paper to cover larger range of nonlinear SDEs when the linear growth condition on the drift coefficient is violated.

1. Introduction

Stochastic differential equations (SDEs) have been widely used in modelling uncertain phenomena in many areas [1,2]. However, due to the difficulty to find general explicit solutions to non-linear SDEs, numerical approximations have been attracting a lot of attention in recent decades [3,4]. One aspect of the numerical analyses for SDEs focuses on asymptotic properties of approximations, among which the asymptotic stability particularly has been interesting to many researchers. There are different types of stabilities, and the almost sure stability and the moment stability are the two that have been discussed a lot. We mention some of the works [5–13] and the references therein. Briefly, those two stabilities are defined by that for any given initial value the solution will decay to the trivial solution (in the sense of moment or almost surely) as time tends to infinity.

However, those stabilities mentioned above sometimes are too strong. In some cases, the solution will not decay to the trivial solution but oscillate as time advances. In this situation, the underlying solution may have a stationary distribution. Stationary distribution of SDEs has many modelling applications, for example in the dynamic of species population [14] and in epidemiology [15]. One way to find the stationary distribution is by solving the Kolmogorov–Fokker–Planck equation. But this is nontrivial. Another way is to approximate it using the stationary distribution obtained from some numerical solution. To follow this approach, one first needs to show the existence and uniqueness of the stationary distribution for the numerical solution. Then the numerical stationary distribution needs to be shown to converge to the underlying one.

The second author's series papers [16–18] are devoted to numerical stationary distributions of stochastic differential equations. In those series papers, the explicit Euler–Maruyama (EM) method was used due to the simple structure and moderate computational cost [19]. However, the explicit EM method has its own restriction, as mentioned in [20], it may not converge to the true solution of the super-linear-coefficient SDEs even in finite time. Therefore, both the drift coefficient

* Corresponding author. Tel.: +44 0 1509 223483.
E-mail addresses: w.liu@strath.ac.uk, w.liu@lboro.ac.uk, lwbvb@hotmail.com (W. Liu), x.mao@strath.ac.uk (X. Mao).

http://dx.doi.org/10.1016/j.cam.2014.08.019

and the diffusion coefficient were required to be global Lipschitz in the series papers. Those restrictions exclude many highly non-linear models, for example [21,15,22] and the references therein.

In this paper, we propose the Backward Euler–Maruyama (BEM) method as the approximation. The BEM method, which is a drift implicit scheme, has been broadly investigated and shown better at dealing with the highly non-linear SDEs in both finite time convergence problems and asymptotic problems. We mention some works [23,24,10,25,11,12,26] here and the references therein. In this paper, we are going to investigate the existence and uniqueness of the numerical stationary distribution of the BEM method and the convergence of it to the underlying stationary distribution. One of our key contributions is that we release the global Lipschitz condition on the drift coefficient by assuming the one-sided Lipschitz condition instead, but we still require the global Lipschitz condition on the diffusion coefficient. And this restriction is due to the techniques employed in the proofs in Section 3, in which the diffusion coefficient needs to be bounded by some linear term. We mention that some papers on the finite time convergence discussed certain type of SDE models with the non-global Lipschitz diffusion coefficient [26]. Therefore, one of the open problems is that can we use some other methods to approximate the stationary distributions of some classes of SDE models without the global Lipschitz on the diffusion coefficient?

This paper is constructed as follows. We first brief the method, definitions, conditions on the SDEs as well as other mathematical preliminaries in Section 2. Then, we propose the coefficients related sufficient conditions for the existence and uniqueness of the numerical stationary distribution in Section 3.1. Under the same conditions, the stationary distribution of the underlying solution is presented in Section 3.2. The convergence of the numerical stationary distribution is proved in Section 3.3. In Section 4, we demonstrate the theoretical results by some numerical simulations. We conclude this paper and discuss some future research in Section 5.

2. Mathematical preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^d$. The transpose of a vector or matrix, M, is denoted by M^T and the trace norm of a matrix, M, is denoted by $|M| = \sqrt{trace(M^T M)}$. If M is a square matrix, the smallest and largest eigenvalues of M are denoted by $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$, respectively.

Let $f, g : \mathbb{R}^d \to \mathbb{R}^d$. To keep symbols simple, let $B(t)$ be a scalar Brownian motion. The results in this paper can be extended to the case of multi-dimensional Brownian motions. We consider the d-dimensional stochastic differential equation of Itô type

$$dx(t) = f(x(t))dt + g(x(t))dB(t) \tag{2.1}$$

with initial value $x(0) = x_0$.

We first assume that the drift coefficient satisfies the local Lipschitz condition and the diffusion coefficient satisfies the global Lipschitz condition.

Condition 2.1. *For any $h > 0$, there exists a constant $C_h > 0$ such that*

$$|f(x) - f(y)|^2 \leq C_h |x - y|^2,$$

for any $x, y \in \mathbb{R}^d$ with $\max(|x|, |y|) \leq h$.

Condition 2.2. *There exists a constant $\bar{K_2} > 0$ such that*

$$|g(x) - g(y)|^2 \leq \bar{K_2} |x - y|^2,$$

for any $x, y \in \mathbb{R}^d$.

We further impose the following condition on the drift coefficient.

Condition 2.3. *Assume there exist a symmetric positive-definite matrix $Q \in \mathbb{R}^{d\times d}$ and a constant $\bar{K_1} \in \mathbb{R}$ such that*

$$(x - y)^T Q (f(x) - f(y)) \leq \bar{K_1} (x - y)^T Q (x - y),$$

for any $x, y \in \mathbb{R}^d$.

From Conditions 2.2 and 2.3, it is easy to see that for any $x \in \mathbb{R}^d$

$$x^T Q f(x) \leq K_1 x^T Q x + \alpha_1, \tag{2.2}$$

and

$$|g(x)|^2 \leq K_2 |x|^2 + \alpha_2, \tag{2.3}$$

with $K_2, \alpha_1, \alpha_2 > 0$ and $K_1 \in \mathbb{R}$.