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On linearized coupling conditions for a class of isentropic multiphase drift-flux models at pipe-to-pipe intersections

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ABSTRACT

In this paper a general drift-flux model describing a subsonic and isentropic multi-phase fluid in connected pipes is considered. Each phase is assumed to be isentropic with its own sonic speed. The components are gamma-law gases with $\gamma > 1$. For such, a computational challenge at a junction is the computation of rarefaction waves which do not have a readily available analytical form. Firstly, the well-posedness of the Riemann problem at the junction is discussed. It is suggested that rarefaction waves should be linearized in order to obtain a more efficient numerical method for coupling such multi-component flow. Some computational results on the dynamics of the multi-phase gas in the pipes demonstrate the qualitative behavior of this approach.

1. Introduction

The isothermal no-slip drift flux model for multiphase flow in a network of pipes can be represented in the form

$$\begin{aligned} &\partial_t \rho_1 + \partial_x \rho_1 u = 0 \\ &\partial_t \rho_2 + \partial_x \rho_2 u = 0 \\ &\partial_t (\rho_1 u + \rho_2 u) + \partial_x \left((\rho_1 + \rho_2) u^2 + p(\rho_1, \rho_2)\right) = 0 \end{aligned} \tag{1}$$

where ρ_1 and ρ_2 are the density of phase 1 and 2, respectively, u is the common velocity and $p = p(\rho_1, \rho_2)$ is the pressure. The two fluids are assumed to be immiscible and, therefore, denote the density of each component as $\rho_1 = \alpha_1 \varrho_1$ and $\rho_2 = \alpha_2 \varrho_2$ with $\alpha_1 + \alpha_2 = 1$ where α_1 and α_2 are the volume fractions of each phase and ϱ_1, ϱ_1 are the phase densities. This flow model is derived from the *two-fluid* model by averaging the balance law for the momentum in the canonical form.

In this paper, the flow of a class of isentropic drift-flux models (1) in a pipe-to-pipe intersection is investigated. Here, every connected pipe is viewed as an arc oriented by parameterization. The focus is on coupling conditions for (1) with a *general pressure law* especially at the junctions of a network giving rise to suitable boundary conditions for the pipes intersecting at a junction where extra conditions are defined for the evolution of the flow solution in the network. A similar study for $p(\rho_1, \rho_2) = a^2 (\rho_1 + \rho_2)$ has been conducted in [1] and for $p(\rho_1, \rho_2) = a_1^2 \rho_1 + a_2^2 \rho_2$ in [2]. We present a theoretical as well as a computational treatment of the general case (with numerical examples for a γ—law) giving rise to the numerical problems of not having an explicit formula for the waves curves. To overcome this difficulty and to obtain an efficient

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numerical integration at the pipe-to-pipe intersection, we linearize the waves curves and solve the coupling conditions on the linearized wave curves. This is the main contribution of this paper. Numerical tests are undertaken which justify this approach. Comparisons are also done with a standard solver. The results compare favorably. As expected, application of the approach to isentropic models gives very good results.

The mathematical study of flow of fluid in networks of pipes is a very active field of research. The interested reader may refer to [3–7] for the case of gas networks, to [8–10] for water networks and [11,12] for traffic networks. These models consider single phase flow. The study of the multiphase case has been introduced in [1,2] for some special drift-flux models where the pressure law was taken as a linear function of the densities allowing for an explicit computation of the wave curves.

The rest of the paper is organized as follows. In Section 2, a brief discussion of the derivation of the flow model is presented. A solution of the standard Riemann problem is given. A mathematical analysis for a junction of pipes in a network is considered in Section 3. A rigorous definition of the Riemann problem at the junction is given including precise coupling conditions. The well-posedness of the Riemann problem at a junction of two connected pipes is presented. The general case of a junction with m ingoing pipes and p outgoing pipes is briefly discussed. Section 4 is devoted to some numerical simulations and results. The effect of the sound speed of the two phases on the junction is investigated. Thereafter the numerical results of the linearization of the Lax curves are presented. Applications of this approach for solving the Riemann problem at the junction of three connected pipes as well as a small network are also presented. A conclusion and ideas for further work are presented in Section 5.

2. Model formulation and preliminary results

A mixture of two fluids with density, volume fraction, velocity and pressure denoted by $\varrho_i,\ \alpha_i,\ u_i,\ p_i$, respectively, for $i = 1, 2$ is considered. A common model for such multiphase fluid is the drift-flux model [13] which takes the form presented in Eq. (1). As in [14], we assume that the pressure and the velocity of the two components are equal. A pressure law $p(\rho_1, \rho_2)$ can be derived as follows: For each component the pressure in this case has the form $p = \hat{\kappa}\rho^\gamma \equiv P(\rho)$ where $\hat{\kappa} \geq 0$ and $\gamma > 1$ are constants. Physically, $P'(\rho) > 0$ for $\rho > 0$. Hence for each phase the pressure takes the form

$$p = p_i(\varrho_i) = a_i^2 \varrho_i^\gamma, \quad i \in \{1, 2\}, \tag{2}$$

where the positive constant a_i is the compressibility factor or sound speed of phase i. The relation $\alpha_1 + \alpha_2 = 1$ can then be written as

$$\frac{\rho_1}{\varrho_1} + \frac{\rho_2}{\varrho_2} = \frac{\rho_1}{\left(p/a_1^2\right)^{1/\gamma}} + \frac{\rho_2}{\left(p/a_2^2\right)^{1/\gamma}} = 1.$$

Hence,

$$p(\rho_1, \rho_2) = (\rho_1 a_1^{2/\gamma} + \rho_2 a_2^{2/\gamma})^\gamma. \tag{3}$$

Clearly, the expression simplifies to the ones studied in [1,2] when the two fluids have the same compressibility factors, $a_1 = a_2 = a$:

$$p(\rho_1, \rho_2) = a^2(\rho_1 + \rho_2)^\gamma. \tag{4}$$

For a mathematical study, the following reformulation of the model is advantageous. Introduce the total momentum and densities

$$I = \hat{\rho} u \quad \text{and} \quad \hat{\rho} = \rho_1 + \rho_2,$$

respectively. Then, (1) is equivalent to the following equation in conservative variables (ρ_1, ρ_2, I) and unspecified pressure $p = p(\rho_1, \rho_2)$

$$\begin{aligned} &\partial_t \rho_1 + \partial_x \frac{\rho_1 I}{\hat{\rho}} = 0; \\ &\partial_t \rho_2 + \partial_x \frac{\rho_2 I}{\hat{\rho}} = 0; \\ &\partial_t I + \partial_x \left(\frac{I^2}{\hat{\rho}} + p\right) = 0. \end{aligned} \tag{5}$$

Before discussing in details the dynamics at a pipe-to-pipe intersection, the solutions of the standard Riemann problem for Eq. (5) are presented. Those will be used later in the construction of solutions to the coupled problem. As it is well known [15], the exact solution to a Riemann problem is constructed as a set of constant states connected by wave curves.

The equation of state $p = p(\rho_1, \rho_2)$ is considered as general as possible and depends on the density of each phase. Later, for illustration and numerical purposes, the isentropic model (3) is used.