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Superconvergent trivariate quadratic spline quasi-interpolants on Worsey–Piper split ☆

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ABSTRACT

In this paper we use Normalized trivariate Worsey–Piper B-splines recently constructed by Sbibih et al. (2012) and the method proposed in Sbibih et al. (2013) to give a new representation of Worsey–Piper Hermite interpolant of any piecewise polynomial of class at least C^1 over the Worsey–Piper split in terms of its polar forms. Using this representation we construct several superconvergent discrete quasi-interpolants. The construction that we present in this work is a generalization of the one presented in Sbibih et al. (2012) with other properties.

1. Introduction

Many applications of splines make use of some approximation method to produce a spline function from given discrete data. Popular methods include interpolation and least squares approximation. However, both of these methods require solution of a linear system of equations with as many unknowns as the dimension of the spline space, and are therefore not suitable for real-time processing of large streams of data. For this purpose local methods, which determine spline coefficients by using only local information, are more suitable. To ensure good approximation properties it is important that the methods reproduce polynomials and preferably the functions in the given spline space.

It is well known that the quasi-interpolation is a general approach to construct a local approximant of a given function f without solving a linear system. Usually, a quasi-interpolant for the function f is obtained as linear combination of some elements of a suitable set of basis functions. In order to achieve local control, these functions are required to be positive, to ensure stability and to have small local supports. The coefficients of the linear combination are the values of linear functionals depending on f and (or) its derivatives or integrals.

In the univariate and bivariate cases, various methods have been developed in the literature (see [1–7], for instance), and many applications are proposed in different fields.

In the trivariate case, the several existing studies for building approximants use the tensor product splines (see [8–13], for instance). In the recent years, G. Nürnberger et al. [14], T. Sorokina et al. [15], C. Rössl et al. [16,17] and M. Rhein et al. [18] proposed some methods to construct quadratic and cubic C^1 spline quasi-interpolants over the uniform type-6 tetrahedral partitions and uniform truncated octahedral partitions. In these methods the Bernstein–Bézier coefficients of the spline quasi-interpolants are immediately available from the given values by applying local averaging. Despite the elegance of these methods, the obtained quasi-interpolants do not reproduce the whole space $\mathbb{P}_2(\mathbb{R}^3)$ of trivariate polynomials of degree less

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than or equal to 2 and they are only second order accurate. In [19], S. Remogna constructed new quasi-interpolation schemes based on the trivariate C^2 quartic box spline defined on a type-6 tetrahedral partition with a fourth approximation order.

In [20], Sbibih et al. considered a regular tetrahedral partition Δ and proposed a normalized B-spline representation of Worsey–Piper (WP) splines of degree 2 and of class C^1 defined in [21]. Such a representation is given in $\mathbb{R}^2$ by Dierckx [22] for C^1 quadratic Powell–Sabin splines and by Speleers [23–25] for bivariate and multivariate Powell–Sabin splines. Sbibih et al. also gave how to express any quadratic polynomial p or WP quadratic spline s as combination of the normalized WP B-splines, where the coefficients in that combination are given in terms of the polar forms of p and s respectively. Then, using these results and the approach given in [6], they constructed several quasi-interpolants which reproduce the space $\mathbb{P}_2(\mathbb{R}^3)$ or the whole space of the WP quadratic splines of class C^1 in $\mathbb{R}^3$.

In numerical analysis, the superconvergence is a phenomenon where the order of convergence of the error estimate, at certain special points, is higher than the order of convergence of the approximant error over the definition's domain. These special points are called superconvergence points. Superconvergence is an important property in theory which can be advantageously exploited in practice. Many superconvergence results have been obtained for multidimensional setting with tensor-product elements, where the superconvergence points are determined from one or two dimensional results by the tensor-product technique (see for instance [26,12,13]). However, when the element is not tensor-product due to either geometry or construction, the problem becomes more complicated.

Recently Sbibih et al. [27] constructed a superconvergent bivariate quadratic quasi-interpolants over Powell–Sabin triangulations. In this work, we want to generalize the results given in [20] using the same method developed in [27]. Then, we give a new trivariate WP B-spline representation of the Hermite interpolant of any piecewise polynomial s of class at least C^1 over a WP split in terms of the polar forms of s. By using this representation, we give a method for constructing superconvergent discrete quasi-interpolants of a sufficiently smooth function f and its first derivatives at the vertices of the tetrahedral partition Δ. We note that the results given in [20] can be obtained, in this work, by taking s of degree 2.

The paper is organized as follows. In Section 2, we recall from [20] some results concerning the construction and the properties of trivariate normalized quadratic B-splines over a Worsey–Piper split of a tetrahedral partition of a domain of $\mathbb{R}^3$. In Section 3, we give the B-spline representation of the Hermite interpolant of all trivariate polynomials of degree $n \geq 2$ or WP splines of degree $n \geq 2$ and of class at least C^1 in terms of their polar forms and the normalized WP B-splines. Using this representation and the approach developed in [6], we construct in Section 4 superconvergent trivariate WP discrete quasi-interpolants. In Section 5, we determine the errors between the function f and the superconvergent discrete quasi-interpolants and between their gradients at the vertices of Δ. Finally, in order to test the performance of our method, we propose in Section 6 some numerical examples.

2. Worsey–Piper splines in $\mathbb{R}^3$

Let Δ be a tetrahedral partition of Ω having vertices V_i with cartesian coordinates (x_i, y_i, z_i), $i = 1, \ldots, N_v$. This means that every pair of domain tetrahedra are disjoint, or share exactly one vertex, one edge, or one face. We denote by Δ^* the Worsey–Piper split of Δ (see [21], for instance), by which each tetrahedron ρ with vertices V_1, V_2, V_3, V_4 is subdivided into twenty-four subtetrahedra in the way shown in Fig. 1. The splitting process is described as follows:

(1) In ρ, select an interior point for each edge, each face, and also one for ρ itself. Let V be the interior point for ρ, denote by W_i the interior point chosen for the face opposite to the vertex V_i and, finally, label the point chosen on the edge joining vertices V_k and V_l by V_{kl}, where $k < l$.
(2) Split the ith face of ρ $(i = 1, \ldots, 4)$ by joining the interior point W_i to the edge points V_{kl} $(k, l \neq i)$, and also to the vertices V_j $(j \neq i)$. This subdivides each of the faces of the tetrahedron into six subtriangles according to the bivariate Powell–Sabin split (see [28]).
(3) Join V, the interior point of ρ, to all points on the boundary, namely, the points W_i, V_j and V_{kl}.

Denote by $S_2^1(\Omega, \Delta^*)$ the linear space of piecewise trivariate quadratic polynomials on Δ^* with C^1 continuity on Ω. Each element of $S_2^1(\Omega, \Delta^*)$ is called a *Worsey–Piper spline (WP spline)*.

In [21], it has been proved that for any set of quadruplet $(f_i, f_{xi}, f_{yi}, f_{zi})$, $i = 1, \ldots, N_v$, the interpolation problem

$$\begin{aligned}&\textit{Find } s \in S_2^1(\Omega, \Delta^*) \textit{ such that}\\ &s(V_i) = f_i, \qquad \frac{\partial s}{\partial x}(V_i) = f_{xi}, \qquad \frac{\partial s}{\partial y}(V_i) = f_{yi}, \qquad \frac{\partial s}{\partial z}(V_i) = f_{zi}, \quad \forall i = 1, \ldots, N_v,\end{aligned} \tag{2.1}$$

has a unique solution $\mathcal{I}_f$ if some geometric constraints are satisfied. More precisely, we have the following result (Theorem 4.3 of [21]).

Theorem 1. *Consider a tessellation of points in $\mathbb{R}^3$ into the tetrahedra $\rho_1, \rho_2, \ldots, \rho_m$, over which are given positional and gradient data at every vertex. In the tessellation, select a unique interior point for each edge, triangle (boundary faces), and tetrahedron. Then, using these points to split each tetrahedron ρ_i $(i = 1, \ldots, m)$ according to the Worsey–Piper split, we can construct a unique C^1 interpolant to the data using piecewise quadratics over the subtetrahedra provided that:*