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Reconstruction of conditional expectations from product moments with applications

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ABSTRACT

In this paper, it is shown under conditions associated with the moment problem that a sequence of product moments uniquely determines a conditional expectation of two random variables. Then, a numerical procedure is derived to reconstruct a conditional expectation in terms of a sequence of its product moments.

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1. Introduction

In this paper, a version of the moment problem is considered. It is shown, if U and W are jointly continuous random variables having a sequence of finite product moments $\alpha_j = E[UW^j]$, $j = 0, 1, 2, \ldots$ satisfying uniqueness conditions associated with a moment problem, then these moments uniquely determine $\psi(w) = E[U \mid W = w]$. Also, a numerical procedure is derived to recover $\psi(w)$ from the α_j. The classical moment problem asks, given a sequence of complex numbers $\alpha_0, \alpha_1, \alpha_2, \ldots$, when does there exist a measure or a function of bounded variation with α_j its jth moment and when is it unique. There are at least three versions of the moment problem, the Hausdorff moment problem, the Stieltjes moment problem and the Hamburger moment problem, all differing by their supports. These classic problems gave rise to an abundance of mathematics, Stieltjes integrals, Pade approximations, orthogonal polynomials, Riesz–Markov theorem. A partial list of famous mathematicians involved in its solution are Chebyshev, Hausdorff, Riesz, Stieltjes, Krein, Markov, Hamburger, Nevanlinna, Akhiezer, Karlin. The texts and tracts [1–3] contain comprehensive coverage of the moment problem.

The Hamburger and Stieltjes moment problems are defined in terms of the Hankel matrices

$$H_n^{(1)} = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n+1} \\ \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_{n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_n & \alpha_{n+1} & \alpha_{n+2} & \cdots & \alpha_{2n} \end{bmatrix} \qquad H_n^{(2)} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n+1} \\ \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_{n+2} \\ \alpha_3 & \alpha_4 & \alpha_5 & \cdots & \alpha_{n+3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n+1} & \alpha_{n+2} & \alpha_{n+3} & \cdots & \alpha_{2n+2} \end{bmatrix}.$$

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The Hamburger moment problem. Suppose an infinite sequence of complex numbers α_j ($j = 0, 1, 2, \ldots$) is given. A necessary and sufficient condition for the existence of a function of bounded variation $\sigma(x)(-\infty < x < \infty)$ with moments α_j satisfying

$$\alpha_j = \int_{-\infty}^{\infty} x^j \sigma(dx) \quad j = 0, 1, 2, \ldots$$

is that the sequence of α_j is positive definite, i.e. $\sum_{j,k\geq 0} \alpha_{j+k} c_j c_k \geq 0$ for an arbitrary sequence $\{c_j,\ j \geq 0\}$, or equivalently $\det(H_n^{(1)}) > 0,\ n = 0, 1, 2, \ldots$. If in addition, there exist constants $C > 0$ and $R > 0$ such that

$$|\alpha_j| \leq CR^j j! \quad j = 1, 2, \ldots$$

then $\sigma(x)$ is unique.

Stieltjes moment problem. Suppose an infinite sequence of complex numbers α_j ($j = 0, 1, 2, \ldots$) is given. Necessary and sufficient conditions for the existence of a function of bounded variation $\sigma(x)(0 < x < \infty)$ with moments α_j satisfying

$$\alpha_j = \int_{0}^{\infty} x^j \sigma(dx) \quad j = 0, 1, 2, \ldots$$

are that the sequences $\det(H_n^{(1)}) > 0$ and $\det(H_n^{(2)}) > 0,\ n = 0, 1, 2, \ldots$. The function $\sigma(x)$ is unique if there exist constants $C > 0$ and $R > 0$ such that

$$|\alpha_j| \leq CR^j (2j)! \quad j = 1, 2, \ldots.$$

Hausdorff moment problem. Suppose an infinite sequence of complex numbers α_j ($j = 0, 1, 2, \ldots$) is given, then a necessary and sufficient condition for the existence of a function of bounded variation, $\sigma(x)$ with finite support $-\infty < a < x < b < \infty$ satisfying

$$\alpha_j = \int_{a}^{b} x^j \sigma(dx) \quad j = 0, 1, 2, \ldots$$

is that the sequence of moments $\alpha_j,\ j = 0, 1, 2, \ldots$ satisfy the inequalities

$$\Delta^k \alpha_j = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \alpha_{i+j} \geq 0, \quad \text{for all } k, j = 0, 1, 2, \ldots. \tag{1}$$

If a solution exists, then it is unique.

There are distributions not uniquely determined by their moments. For example, Stieltjes [4,5] showed distributions with densities

$$a \exp(-t^{1/4}) \quad \text{and} \quad b_k t^{k-\log t} \quad t \in (0, \infty)$$

are not uniquely determined by their moments, where a, b_k are positive normalizing constants. Heyde [6] showed that the familiar lognormal distribution is not moment determinate. Other examples can be found in [7,8]. Common terminology calls a distribution uniquely determined by its moments M-determinate, if not, it is called M-indeterminate. The lognormal is an M-indeterminate distribution. If $F(w)$ is the distribution function of the random variable W that is M-indeterminate, then setting $U = F(W)$

$$E[U \mid W = w] = F(w)$$

makes $E[U \mid W = w]$ M-indeterminate in terms of $\sigma(x) = \int_0^x F(w)dF(w) = F^2(x)/2$. This follows from the Krein condition: $F(x)$ is M-indeterminate if $\int_{-\infty}^{\infty} -\ln f(x)/(1+x^2)dx < \infty$, $f(x)$ the probability density of $F(x)$, $f(x) > 0$ for all x. Criteria other than those stated in the moment problems above can be used to determine if a distribution is M-determinate. There are the Carleman and Krein conditions reviewed in [9]. The criteria in the Krein condition is pleasing, because it depends entirely on the probability density function of $F(x)$.

Except for trivial text book examples, an exact expression for $E[U \mid W = w]$ is rare. The Gaussian case is an exception, i.e. when U and W are jointly normal, the conditional expectation is exactly known to be linear. For the Hausdorff case, it is shown in [10] how to recover a distribution function in terms of its moments. Recovery of a distribution function from moments for the Hausdorff moment problem can be found in [11–15]. We derive an expression for $E[U \mid W = w]$ in terms of $\alpha_j, j = 0, 1, 2, \ldots$. More precisely, we show in Theorem 2 under certain assumptions

$$E[U \mid W = w] = \lim_{n\to\infty} \frac{1}{p(w)} \sum_{j=0}^{2n} v_{[n(M_n+w)/M_n]j} \frac{n^{j+1}}{M_n^{j+1}} \alpha_j$$

where v_{ij} are entries in a Vandermonde matrix, $p(w)$ is the marginal density of W and $M_n/n \to 0$.

Numerous applications exist either in stochastic processes, economics, statistics, control theory and other branches of applied mathematics where an estimate for $E[U \mid W]$ is needed. Because conditional expectations are projections, they are important for deriving optimal estimators. This concept is particularly important in regression and filtering type problems. Applications in structural equations in economics, the Kalman–Bucy Filter in control theory and in regression in statistics can be found in [16–18]. Our solution, is applied to the errors in variables regression problem, see [19] for details on errors in variables.