Journal of Computational and Applied Mathematics 278 (2015) 29–36

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Numerical solution of Volterra–Fredholm integral equations using Legendre collocation method

S. Nemati *

Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran

ARTICLE INFO

Article history:
Received 16 April 2014
Received in revised form 30 July 2014

MSC:
65R20
42C10

Keywords:
Volterra–Fredholm integral equations
Shifted Legendre polynomials
Gauss–Legendre nodes
Matrix equation

ABSTRACT

A numerical method for solving the Volterra–Fredholm integral equations is presented. The method is based upon shifted Legendre polynomials approximation. The properties of shifted Legendre polynomials are first presented. These properties together with the shifted Gauss–Legendre nodes are then utilized to reduce the Volterra–Fredholm integral equations to the solution of a matrix equation. An estimation of the error is given. Illustrative examples are included to demonstrate the validity and applicability of the technique.

1. Introduction

One of the fundamental classes of equations is integral equations. Such equations arise in many areas of science and engineering (see, [1–3]) and play an important role in the modeling of real-life phenomena in other fields of science (see, [4]). For these reasons, integral equations have received much attention in the last decades. Some of integral equations cannot be solved by the well-known exact methods. Hence, it is desirable to introduce numerical methods with high accuracy to solve these equations numerically.

Over the years, it was found that the spectral methods are a valid method to obtain approximations for integral equations (see for example, [5–11]). Our aim in this paper is to propose a method to approximate a class of Volterra–Fredholm integral equations on the interval $[0, l]$ by using the shifted Legendre polynomials.

Consider the following Volterra–Fredholm integral equation

$$A(x)y(x) + B(x)y(h(x)) = f(x) + \lambda_1 \int_0^{h(x)} k_1(x,t)y(t)dt + \lambda_2 \int_0^{l} k_2(x,t)y(h(t))dt \tag{1}$$

where the functions $k_1(x, t)$ and $k_2(x, t)$ are known kernel functions on the interval $[0, l] \times [0, l]$ and the functions $A(x), B(x), h(x)$ and $f(x)$ are known functions defined on the interval $[0, l]$ and $0 \le h(x) < \infty$, $y(x)$ is the unknown function and λ_1, λ_2 are real constants such that ${\lambda_1}^2 + {\lambda_2}^2 \neq 0$. When $h(x)$ is a first-order polynomial, Eq. (1) is called functional integral equation with proportional delay. Numerical solution of this class of integral equations has been introduced using Lagrange collocation method by K. Wang and Q. Wang in [12]. Also, they have applied Taylor collocation method to solve

* Tel.: +98 9128058550.
E-mail addresses: s.nemati@umz.ac.ir, somayeh.nemati82@gmail.com.

http://dx.doi.org/10.1016/j.cam.2014.09.030
0377-0427/

Eq. (1) numerically. In this paper, we will apply the shifted Legendre collocation method to approximate the solution of Eq. (1).

The paper is organized as follows: Preliminaries needed hereafter are given in Section 2. In Section 3, we design the shifted Legendre collocation technique to solve Eq. (1). An estimation of the error is provided in Section 4, when a sufficiently smooth function is expanded in terms of the shifted Legendre polynomials. In Section 5, we present three numerical examples to illustrate the validity and applicability of the method. Concluding remarks are given in Section 6.

2. Preliminaries and notations

The well-known Legendre polynomials are defined on the interval $[-1, 1]$ and can be determined with the aid of the following recurrence formulae:

$$L_0(x) = 1,$$
$$L_1(x) = x,$$
$$L_{m+1}(x) = \frac{2m+1}{m+1}xL_m(x) - \frac{m}{m+1}L_{m-1}(x), \quad m = 1, 2, \dots, \quad -1 \le x \le 1.$$

In order to use these polynomials on the interval $[0, l]$ we define the so-called shifted Legendre polynomials of degree i as follows:

$$\psi_i(x) = L_i\left(\frac{2}{l}x - 1\right), \quad i = 0, 1, 2, \dots.$$

We consider the space $L^2[0, l]$ equipped with the following inner product and norm:

$$\langle f, g\rangle = \int_0^l f(x)g(x)dx, \qquad \|y\|_2 = \langle y, y\rangle^{\frac{1}{2}}.$$

The set of shifted Legendre polynomials forms a complete $L^2[0, l]$-orthogonal system such that the orthogonality condition is

$$\int_0^l \psi_i(x)\psi_j(x)dx = \begin{cases} \dfrac{l}{2i+1}, & i = j, \\ 0, & i \ne j. \end{cases}$$

A function $y(x) \in L^2[0, l]$, may be expressed in terms of shifted Legendre polynomials as

$$y(x) = \sum_{i=0}^{\infty} c_i\psi_i(x),$$

where the coefficients c_i are given by

$$c_i = \frac{2i+1}{l}\int_0^l y(x)\psi_i(x)dx, \quad i = 0, 1, 2, \dots.$$

Considering only the first $(n + 1)$-terms of shifted Legendre polynomials we have

$$y(x) \simeq \sum_{i=0}^{n} c_i\psi_i(x) = C^T\psi(x),$$

where the shifted Legendre coefficient vector C and the shifted Legendre vector $\psi(x)$ are given by

$$C = [c_0, c_1, \dots, c_n]^T, \tag{2}$$
$$\psi(x) = [\psi_0(x), \psi_1(x), \dots, \psi_n(x)]^T. \tag{3}$$

3. Numerical solution

In this section, we approximate the solution of Eq. (1) using shifted Legendre polynomials. We assume that the known functions in Eq. (1) satisfy the conditions that this equation has a unique solution [5,13–16].

We approximate the function $y(x)$ using the way mentioned in the previous section as follows:

$$y(x) \simeq \sum_{i=0}^{n} c_i\psi_i(x) = C^T\psi(x), \tag{4}$$

where the coefficients $c_i, i = 0, 1, \dots, n$ are the unknowns to be determined and C and $\psi(x)$ are defined as (2) and (3), respectively.