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# New quadrature rules for highly oscillatory integrals with stationary points

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## HIGHLIGHTS

- A new framework for the numerical solution of highly oscillatory integrals is proposed.
- The integrals are bifurcated in the neighborhood of stationary point.
- The integral on the smaller subinterval is solved by hybrid functions and Haar wavelets.
- The integral on the longer subinterval is solved by the meshless method with MQ radial basis functions.
- Convergence analysis of the proposed methods is performed.

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ABSTRACT

In this paper new algorithms are being proposed for evaluation of highly oscillatory integrals (HOIs) with stationary point(s). The algorithms are based on modified Levin quadrature (MLQ) with multiquadric radial basis functions (RBFs) coupled with quadrature rules based on hybrid functions of order 8 (HFQ8) and Haar wavelets quadrature (HWQ) (Aziz et al. 2011). Part of the new procedure presented in this paper is comprised of transplanting monomials (which are used in the conventional Levin method) by the RBFs. The linear and Hermite polynomials based quadratures (Xiang, 2007) are being replaced by the new methods based on HWQ and HFQ8 respectively. Both the methods are merged with MLQ to obtain the numerical solution of highly oscillatory integrals having stationary points. The accuracy of the new methods is neither dampened by presence of the stationary point(s) nor by the large value of frequency parameter $\omega$. Theoretical facts about the error analysis of the new methods are analyzed and proved. Numerical examples are included to show efficiency and accuracy of the new methods.



## 1. Introduction

Oscillatory integrals occur in many practical applications. These include diffraction theory and optics in physics, quantum chemistry, image analysis, signal processing, electrodynamics, computerized tomography and fluid mechanics. Oscillatory integrals are also very common in the Fourier analysis which are widely used in science and engineering. Closed form solutions of HOIs are rarely available, hence the viable way left is to solve them numerically. Numerical solution of the oscillatory integrals and the integrals having stationary points is challenging for the conventional numerical methods like

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trapezoidal rule, Simpson rule and Gauss–Legendre quadrature etc. These methods become ineffective and computationally intensive when used for numerical solution of the HOIs. The inherent reason for the failure of these methods is that in the case of high frequency integrands, the quadrature points cannot match the frequency of the oscillations.

In the literature, a few numerical algorithms are reported which are specially designed for numerical solution of the HOIs. This area is relatively new and is catching the attention of the researchers. Further improvements are needed in the existing algorithms to cope with the challenges arising in the numerical solution of HOIs. In recent years, steady and substantial growth has been observed in this area. Some of the relevant work reported so far in this field include [1–11].

In [4], Levin proposed a novel method for numerical solution of the HOIs. This work has been followed by some more contributions in the form of [12,2,13,14], which are being aimed at the numerical solution of non-oscillatory integrals and HOIs with and without stationary point. These methods, if correctly utilized can exhibit good accuracy and convergence with less computational cost for larger values of the frequency parameter.

The Filon-type of methods [7–9,5,10,11,15] which are meant for efficient evaluation of the HOIs in the asymptotic sense. These methods are efficient, if the $k$-moments, $I(x^k) = \int_a^b x^k e^{i\omega g(x)}\mathrm{d}x$ are easily computable, which is sometimes counted as the limitations of these methods as well. Similarly, numerical methods designed for HOIs which are based on asymptotic techniques like the method of stationary phase, the method of steepest descent and related saddle point methods are the part and parcel of existing literature.

In this paper, we will consider the standard form of one-dimensional highly oscillatory integral (HOI) defined on the interval $[a, b]$ as:

$$I = \int_a^b f(x)e^{i\omega g(x)}\mathrm{d}x, \tag{1}$$

where the function $g(x)$ has necessarily critical point(s) in the interval $[a, b]$. In the above integral, the parameter $\omega$ is the frequency parameter whose large value makes the conventional methods ineffective. The functions $f$ and $g$ are assumed to be non-oscillatory and smooth functions, often called the amplitude and the phase functions of the integral respectively.

Recently, the author in [6] has proposed a more sophisticated Levin-type hybrid method for obtaining an accurate numerical solution of HOIs having stationary points. In the paper [6], a Levin's type of collocation method is coupled with Gaussian quadrature having linear and Hermite basis to solve HOIs containing stationary point(s). The domain interval is bifurcated into two subintervals, the one containing stationary point(s) is the smaller interval and where numerical solution is found by Gaussian quadrature. In the remaining subinterval (the bigger one), which is free from stationary point(s), the numerical solution is obtained by the conventional Levin type method with the monomial basis [4]. The proposed methods are being accompanied by the detailed error analysis as well.

In the present work we retain the same principle of interval subdivision as proposed in [6]. Improvements in accuracy can be achieved if the quadrature based on linear and Hermite polynomials are replaced by new quadrature rules HWQ and HFQ8 [2,12,13], respectively. Similarly, good accuracy can be obtained if the conventional Levin quadrature is replaced by MLQ [2,13]. HFQ8 approximates the integral on the sub-interval that contains stationary point(s) and MLQ is used to approximate the integral on the sub-interval that contains no stationary point(s). For the sake of simplicity, we assume that $x = 0$ is the unique critical point of the integrand, that lies inside or at the end point of the interval $[a, b]$ and $\omega \gg 1$.

In the present paper we have found out the theoretical error bounds for our methods based on hybrid functions and Haar wavelets which were not explored in our earlier papers [2,12,13]. These findings have been presented in the form of Theorem 1. The error bound derived for the hybrid and Haar based algorithms are used in the error bound of the meshless method. The new contribution is presented in the form of Theorem 2.

The rest of the paper is organized as follows. In Section 2 the two methods are described along with error analysis. In Section 3, numerical results and analysis about the performance of the methods are included. The paper contains some conclusions at the end.

## 2. Implementation procedure and error analysis

In this section we discuss the implementation procedure and the error bounds of HFQ8, HW and MLQ. The theoretical results are summarized in the following sub-sections.

### 2.1. Hybrid functions

The orthogonal set of hybrid functions $\psi_{ij}(x),\ i = 1, 2, \ldots, n$ and $j = 0, 1, \ldots, m-1$ is defined on the interval $[0, 1)$ as

$$\psi_{ij}(x) = \begin{cases} L_j(2nx - 2i + 1), & \text{for } x \in \left[\dfrac{i-1}{n}, \dfrac{i}{n}\right) \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$