Journal of Computational and Applied Mathematics 278 (2015) 90–100

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

# Smoothness and error bounds of Martensen splines

Vittoria Demichelis *, Matteo Sciarra

*Department of Mathematics "G. Peano", University of Torino, Via Carlo Alberto, 10, I-10123 Torino, Italy*

ARTICLE INFO

*Article history:*
Received 1 April 2014
Received in revised form 15 July 2014

*MSC:*
65D07
65D05

*Keywords:*
Polynomial spline
Hermite interpolation

ABSTRACT

Martensen splines $Mf$ of degree $n$ interpolate $f$ and its derivatives up to the order $n-1$ at a subset of the knots of the spline space, have local support and exactly reproduce both polynomials and splines of degree $\leq n$. An approximation error estimate has been provided for $f \in \mathbb{C}^{n+1}$.

This paper aims to clarify how well the Martensen splines $Mf$ approximate smooth functions on compact intervals. Assuming that $f \in \mathbb{C}^{n-1}$, approximation error estimates are provided for $D^j f$, $j = 0, 1, \ldots, n-1$, where $D^j$ is the $j$th derivative operator. Moreover, a set of sufficient conditions on the sequence of meshes are derived for the uniform convergence of $D^j Mf$ to $D^j f$, for $j = 0, 1, \ldots, n-1$.



## 1. Introduction

In the construction of spline approximation operators it is desirable to obtain the three properties of locality, interpolation and optimal polynomial reproduction. However, when the knots of the spline space are chosen to coincide with interpolation points, the properties of locality and interpolation are incompatible for quadratic or higher degree splines [1].

Using a procedure based on the introduction of additional knots, De Villiers and Rohwer [1] constructed, for arbitrary order, an optimal nodal spline interpolation operator possessing the three desired properties.

This idea was introduced by De Villiers and Rohwer [1] as an alternative to quasi-interpolation methods and generalized by Dahmen, Goodman and Micchelli in [2], where the authors studied minimal Hermite spline interpolation which was first investigated by Martensen [3].

Considering the two point Hermite spline interpolation scheme studied by Martensen [3], Siewer [4,5] constructed the Martensen splines $Mf$ of order $n+1$ (degree $\leq n$) obtaining the properties of locality, interpolation of $f$ and its derivatives up to the order $n-1$ at a subset of the knots of the spline space and optimal polynomials and splines reproduction. Approximation properties of $Mf$ have been considered in [4], where an error estimate has been provided for $f \in \mathbb{C}^{n+1}$. Martensen splines in the case of equidistant knots and bivariate constructions using Boolean methods have been studied respectively in [6] and [7].

In the present paper, we will continue the investigation of Siewer on how well the Martensen splines $Mf$ approximate smooth functions $f$ on compact intervals. Assuming that $f \in \mathbb{C}^{n-1}$, we shall provide approximation error estimates for $D^j f$, $j = 0, 1, \ldots, n-1$, where $D^j$ is the $j$th derivative operator. Moreover, we shall give a set of sufficient conditions on the sequence of spline knots for the uniform convergence of $D^j Mf$ to $D^j f$ for $j = 0, 1, \ldots, n-1$. In virtue of their approximation properties, Martensen splines can be used for the numerical evaluation of certain finite-part integrals [8–10].

* Corresponding author. Tel.: +39 011 6702815; fax: +39 011 6702878.
*E-mail address:* vittoria.demichelis@unito.it (V. Demichelis).

http://dx.doi.org/10.1016/j.cam.2014.09.027
0377-0427/

## 2. Martensen splines approximation

In this section, we give the necessary background material on Martensen splines based on the works in [4,5].

Let $T = \{t_j\}_{j\in\mathbb{Z}}$ be a strictly increasing sequence of points in $\mathbb{R}$. We write $\mathbb{P}^n$ for the set of polynomials of degree $n$ and $S_{n+1}(T)$ for the set of polynomial splines of order $n+1$, with simple knots at the points $t_j$, so that $S_{n+1}(T) \subset \mathbb{C}^{n-1}(\mathbb{R})$.

We denote by $\{B_{j,m}(x)\}_{j\in\mathbb{Z}}$ the set of normalized $B$-splines of order $m$ on $T$, having support $[t_j, t_{j+m}]$ and defined by [11]

$$B_{j,m}(x) := (-1)^m (t_{j+m} - t_j)[t_j, \dots, t_{j+m}](x - \cdot)_+^{m-1},$$

where the symbol $[t_j, \dots, t_{j+m}]$ denotes the $m$th-order divided-difference functional and

$$x_+^r = \begin{cases} x^r, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

The following theorem, stated and proved in [3], specifies the Martensen interpolation scheme.

**Theorem 1** ([3]). *Let $\alpha_0^k,\ \alpha_n^k$, with $k = 0, \dots, n-1$, be arbitrarily given real numbers. For $n \in \mathbb{N}$ and the set of knots $T_n = \{\alpha = t_0 < t_1 < \dots < t_n = \beta\}$, there is a uniquely determined spline $H_n(t),\ t \in [\alpha, \beta]$, with $H_n \in S_{n+1}(T_n)$, satisfying*

$$D^k H_n(t_0) = \alpha_0^k, \quad k = 0, \dots, n-1,$$
$$D^k H_n(t_n) = \alpha_n^k, \quad k = 0, \dots, n-1.$$

A generalization of this theorem has been obtained by considering nonsymmetrical interpolation conditions [2].

For $n \in \mathbb{N}, j \in \mathbb{Z}$ and $0 \leq i < n$, Siewer [4] provides a constructive proof for Theorem 1 by defining recursively the fundamental Hermite splines $G_{i,n,t_j,\dots,t_{j+n}}(x) \in S_{n+1}(T)$ and $H_{i,n,t_j,\dots,t_{j+n}}(x) \in S_{n+1}(T)$ satisfying

$$D^k G_{i,n,t_j,\dots,t_{j+n}}(t_j) = 0, \quad i, k = 0, \dots, n-1,$$
$$D^k G_{i,n,t_j,\dots,t_{j+n}}(t_{j+n}) = \delta_{i,k}, \quad i, k = 0, \dots, n-1,$$

and

$$D^k H_{i,n,t_j,\dots,t_{j+n}}(t_j) = \delta_{i,k}, \quad i, k = 0, \dots, n-1,$$
$$D^k H_{i,n,t_j,\dots,t_{j+n}}(t_{j+n}) = 0, \quad i, k = 0, \dots, n-1.$$

Here, we are interested in the representation of $G_{i,n,t_j,\dots,t_{j+n}}$ and $H_{i,n,t_j,\dots,t_{j+n}}$ as a linear combination of $B$-splines $B_{s,n+1}$. In order to obtain this expansion, we need the Marsden identity.

**Theorem 2** ([12]). *Given any increasing (not necessarily strictly) knot sequence $T = \{t_j\}_{j\in\mathbb{Z}}$ and two indices $l \leq r$ with $t_l < t_{r+1}$, for all $y \in \mathbb{R}$ and all $x \in [t_l, t_{r+1})$ the following identity holds*

$$(y - x)^n = \sum_{s=l-n}^{r} \varphi_{s,n}(y) B_{s,n+1}(x), \tag{1}$$

*where*

$$\varphi_{s,n}(y) := \prod_{\nu=1}^{n} (y - t_{s+\nu}) \in \mathbb{P}^n \tag{2}$$

*is the dual polynomial for $B_{s,n+1}$.*

Taking in account the strict monotonicity of the considered knot sequence $T$, the following $B$-Spline expansion for fundamental Hermite splines is provided in [4].

**Theorem 3** ([4]). *Let $n \in \mathbb{N}$ and $i \in \{1, \dots, n\}$. For $x \in [t_j, t_{j+n}], j \in \mathbb{Z}$, the fundamental Hermite splines for the Martensen interpolation allow the B-Spline expansion*

$$G_{n-i,n,t_j,\dots,t_{j+n}}(x) = \sum_{s=j}^{j+n-1} \frac{(-1)^{n-i}}{n!} \frac{d^i}{dy^i} \varphi_{s,n}(t_{j+n}) B_{s,n+1}(x) \tag{3}$$

*and*

$$H_{n-i,n,t_j,\dots,t_{j+n}}(x) = \sum_{s=j-n}^{j-1} \frac{(-1)^{n-i}}{n!} \frac{d^i}{dy^i} \varphi_{s,n}(t_j) B_{s,n+1}(x). \tag{4}$$