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On an optimization problem related to static super-replicating strategies ☆

Xinliang Chen [a], Griselda Deelstra [b], Jan Dhaene [c], Daniël Linders [c,*], Michèle Vanmaele [d]

[a] ING, Brussels, Belgium
[b] Université Libre de Bruxelles, Brussels, Belgium
[c] KU Leuven, Leuven, Belgium
[d] Ghent University, Gent, Belgium

HIGHLIGHTS

- We investigate three issues related to super-replicating strategies for options written on a weighted sum of asset prices.
- The first issue is the (non-)uniqueness of the optimal solution.
- The second issue is the generalization to an optimization problem where the weights may be random.
- The third issue is the study of the co-existence of the comonotonicity property and the martingale property.

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ABSTRACT

In this paper, we investigate an optimization problem related to super-replicating strategies for European-type call options written on a weighted sum of asset prices, following the initial approach in Chen et al. (2008). Three issues are investigated. The first issue is the (non-)uniqueness of the optimal solution. The second issue is the generalization to an optimization problem where the weights may be random. This theory is then applied to static super-replication strategies for some exotic options in a stochastic interest rate setting. The third issue is the study of the co-existence of the comonotonicity property and the martingale property.

1. Introduction

Self-financing portfolios play an important role in hedging, trading and valuation. When a self-financing portfolio dominates an exotic option in terms of its pay-off, it is a super-replicating portfolio. In addition, when the weights of the elements in a super-replicating portfolio are fixed from the starting time, it is a static super-replicating portfolio, and the corresponding strategy is called a static super-replicating strategy.

For a given exotic option, in general, several strategies will exist which super-replicate its pay-off. One of the aims of this paper is to investigate the problem of finding the cheapest strategy in a well-defined class of admissible super-replicating strategies for the exotic option under consideration.

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* Corresponding author.
E-mail addresses: xinliang.chen@hotmail.com (X. Chen), griselda.deelstra@ulb.ac.be (G. Deelstra), Jan.Dhaene@kuleuven.be (J. Dhaene), Daniel.Linders@kuleuven.be (D. Linders), michele.vanmaele@UGent.be (M. Vanmaele).

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1.1. Static super-replicating strategies

For $i = 1, 2, \dots, n$, the random variable X_i, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ denotes the price of an asset at some future date $T_i,\ 0 \le T_i \le T$. Hereafter, we always assume that all X_i are positive r.v.'s.[1] The current time-0 price of a European call option with pay-off $(X_i - K)_+$ at maturity T_i is denoted by $C_i[K]$. We assume that these options are traded on an options exchange and we can observe the market prices for these options.

Chen et al. [1] consider a class of European call type exotic options written on $S = \sum_{i=1}^n w_i X_i$ for some deterministic weights $w_i > 0$, which have a pay-off at expiration time T equal to $(S - K)_+$. The inequality

$$(S-K)_+ = \left(\sum_{i=1}^n w_i X_i - K\right)_+ \le \sum_{i=1}^n w_i (X_i - K_i)_+, \quad \mathbb{P}\text{-a.s.}, \tag{1}$$

always holds for all $(K_1, K_2, \dots, K_n)$ satisfying $\sum_{i=1}^n w_i K_i \le K$ and $K_i \ge 0,\ i = 1, \dots, n$. Static super-replicating strategies with pay-off $\sum_{i=1}^n w_i (X_i - K_i)_+,\ \sum_{i=1}^n w_i K_i \le K$, are studied in [1] in a deterministic interest rate setting.

It has been proven that one can obtain an 'optimal' decomposition $K = \sum_{i=1}^n w_i K_i^*$ with an explicit expression for the optimal $K_i^*,\ i = 1, \dots, n$; see [2] or [3]. A simplified version of it can be found in Theorem 1 of the next section in this paper. Using the optimal decomposition $K_i^*,\ i = 1, \dots, n$, the corresponding super-replicating strategy for the exotic call option has the least price at time zero among a general class of super-replicating investment strategies.

For the moment, we assume that the risk-free rate r is deterministic and constant. In Section 3, we relax this assumption and consider the case where interest rates behave stochastically. From inequality (1) and the discussion above, we find that the optimal super-replicating strategy for an exotic call option consists of buying at time zero $w_i e^{-r(T-T_i)}$ European vanilla call options with pay-off $(X_i - K_i^*)_+$ at time T_i and holding these options until they expire at time T_i. We exercise those options with positive pay-offs and invest the eventual pay-offs at that time in the risk-free account until time T. The time-0 price of this optimal super-replicating strategy is given by

$$\sum_{i=1}^n w_i e^{-r(T-T_i)} C_i\left[K_i^*\right]. \tag{2}$$

The upper bound (2) for the time-0 price of an exotic call as well as the corresponding super-replicating strategy can be obtained in an infinite market case, meaning that prices $C_i[K]$ of vanilla call options are available for all strikes K, and in a finite market case, where only a finite number of vanilla call option prices are observed; see e.g. [4,1]. In [5] it is noticed that in the infinite market case, the cheapest super-replicating strategy for the exotic call option derived above cannot be improved by adding other traded derivatives to the financial market, as long as these derivatives are written on a single asset. Remark that in the finite market case this result does not necessarily hold.

The current time-0 price of a European put option with pay-off $(K - X_i)_+$ at maturity T_i is denoted by $P_i[K]$. Assume now that also vanilla put options are traded in the financial market and consider the exotic put option with pay-off $(K - S)_+$, at time T. The inequality

$$(K-S)_+ = \left(K - \sum_{i=1}^n w_i X_i\right)_+ \le \sum_{i=1}^n w_i (K_i - X_i)_+, \quad \mathbb{P}\text{-a.s.}, \tag{3}$$

holds for all $(K_1, K_2, \dots, K_n)$ satisfying $\sum_{i=1}^n w_i K_i \ge K$ and $K_i \ge 0,\ i = 1, \dots, n$. Similar to inequality (1), one can derive an 'optimal' decomposition $K = \sum_{i=1}^n w_i K_i^*$ with an explicit expression for the optimal $K_i^*,\ i = 1, \dots, n$. Furthermore, we find from (3) that the optimal super-replicating strategy for the exotic put consists of buying a portfolio of European vanilla put options and the time-0 price is given by

$$\sum_{i=1}^n w_i e^{-r(T-T_i)} P_i\left[K_i^*\right]. \tag{4}$$

For more details we refer to Linders et al. [5].

Examples of options with a pay-off at time T equal to $(S - K)_+$ or $(K - S)_+$ are basket options and Asian options. In the case of a basket option, we have that $T_i = T$ and the random variable X_i denotes the price level of stock i at time T, while S is a weighted sum of the stock price levels at time T. In the case of Asian options, only one asset is involved. The random variable X_i represents the price level of this asset at time $T - i + 1$. The weights w_i typically equal $\frac{1}{n}$ such that S is the average price of the asset over the last n periods prior to expiration.

1.2. The optimization problem

Hereafter, we always assume that the financial market is arbitrage-free and that there exists a risk-neutral pricing measure $\mathbb{Q}$, equivalent to the physical measure $\mathbb{P}$, such that the current price of any pay-off can be represented as the discounted

1 Throughout this paper, all random variables are assumed to have finite expectations.