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Asymptotes of space curves ☆

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ABSTRACT

In Blasco and Pérez-Díaz (2014) (see [3]), a method for computing *generalized asymptotes* of a real algebraic plane curve implicitly defined is presented. Generalized asymptotes are curves that describe the status of a branch at points with sufficiently large coordinates and thus, it is an important tool to analyze the behavior at infinity of an algebraic curve. This motivates that in this paper, we analyze and compute the generalized asymptotes of a real algebraic space curve which could be parametrically or implicitly defined. We present an algorithm that is based on the computation of the *infinity branches* (this concept was already introduced for plane curves in Blasco and Pérez-Díaz (2014) [1]). In particular, we show that the computation of infinity branches in the space can be reduced to the computation of infinity branches in the plane and thus, the methods in Blasco and Pérez-Díaz (2014) (see [1]) can be applied.

1. Introduction

In [1], we introduce the notion of *infinity branches* and *approaching curves*. Some important properties are derived from these concepts which allow us to obtain an algorithm that compares the behavior of two implicitly defined algebraic plane curves at infinity. In particular, a characterization for the finiteness of the Hausdorff distance between two algebraic curves in the n-dimensional space can be obtained (see Section 5 in [1], and [2]). The characterization is related with the asymptotic behavior of the two curves and it can be easily checked.

Based on the notions and results presented in [1], in [3] we deal with the problem of computing the asymptotes of the infinity branches of a given plane curve $\mathcal{C}$ implicitly defined. The asymptotes of an infinity branch of $\mathcal{C}$ reflect the status of this branch at points with sufficiently large coordinates. It is well known that an asymptote of a curve is a line such that the distance between the curve and the line approaches zero as they tend to infinity. However, in [3], we show that an algebraic plane curve may have more general curves than lines describing the status of a branch at infinity. Thus, in [3], we develop an algorithm for the computation of *generalized asymptotes* (or *g-asymptotes*), and some important properties concerning this new concept are presented.

The applicability of the results presented in [1,3] is of central importance in the field of computer aided geometric design (CAGD) since these results provide new and important concepts as well as computational techniques that allow us to obtain information about the behavior of a plane curve at infinity. For instance, the infinity branches of an implicit plane curve $\mathcal{C}$ are essential for the study of the topology of $\mathcal{C}$ (see e.g. [4–6]) or for detecting its symmetries (see e.g. [7]). Also, the results obtained play an important role in the frame of approximate parametrization problems (see e.g. [8,9]) or in analyzing the Hausdorff distance between two curves (see [2]) which is specially interesting since the Hausdorff distance is an appropriate tool for measuring the closeness between two curves (see e.g. [10–13]).

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The results obtained and the importance of the applications we mentioned above moved us to try to generalize the foundations and methods in [1,3] to the case of space curves.

For this purpose, in this paper, we consider an irreducible real algebraic space curve $\mathcal{C}$ over the field of complex numbers $\mathbb{C}$ implicitly defined by two irreducible real polynomials, and we deal with the problem of computing the asymptotes of the infinity branches of $\mathcal{C}$. For this purpose, we generalize the notions and some previous results presented in [1,3], and we develop an algorithm for computing the g-asymptotes of $\mathcal{C}$ (see Sections 2–4).

In addition, we also show how to compute the g-asymptotes if the given curve is defined by a rational real parametrization. This parametric approach can be easily generalized for parametric plane curves and in general, for a rational parametrization of a curve in the n-dimensional space (see Section 5). This new statement of the problem is specially interesting in some practical applications in CAGD, where objects are often given and manipulated parametrically.

Authors have not been able to find many references in the literature dealing with the analysis and computation of infinity branches and asymptotes of a given algebraic curve. Only in [14], linear asymptotes of space curves are briefly studied. In particular, it is proved how the tangents at the simple points at infinity of the curve (i.e. non-singular points at infinity) are related with the asymptotes. For the case of plane curves, some results concerning linear asymptotes can be found in [15,16].

CAGD is a natural environment for practical applications of algebraic curves and surfaces. In particular, the results and methods presented in this paper open new ways to study the behavior of algebraic space curves, with expected generalizations to higher dimension and the case of surfaces.

The applications expected of the results obtained in this paper can be included in the frame of those presented for the plane case. More precisely, the methods and techniques developed could be very useful to deal (for instance) with the following problems: the behavior at infinity of a space curve when approximate parametrization techniques are used (see e.g. [14] or [17]), the sketch of the graph or the computation of the topology of real algebraic space curves (see [17–19] or [20]), the detection of the symmetries of a given space curve (see [7]) or the computation of the Hausdorff distance between two curves (see [2,13]). The reader may find explanations of these and other problems in the vast literature on CAGD (see e.g. [20–25]).

The structure of the paper is as follows: in Section 2, we present the notation and we generalize some previous results developed in Sections 2, 3 and 4 in [1]. In particular, we introduce the notions of *infinity branch* and *convergent infinity branches*, and we characterize whether two implicit algebraic space curves approach each other at infinity. In Section 3, we show the relation between infinity branches of plane curves and infinity branches of space curves. More precisely, we obtain the infinity branches of a given space curve $\mathcal{C}$ from the infinity branches of a certain plane curve obtained by projecting $\mathcal{C}$ along some "valid projection direction". This approach allows us to use effective computational techniques existing in the plane case (see Section 3 in [1]) for the computation of the infinity branches of space curves. In Section 4, we introduce the notions of *perfect curve* and *generalized asymptote* or *g-asymptote*. These concepts are derived from the study of approaching curves and convergent branches in Section 3, and they generalize the notions introduced in [3] (see Section 3) for a given plane curve. Moreover, in Section 4, we also present an efficient algorithm that computes a g-asymptote for each infinity branch of a given space curve implicitly defined. We reach the expected situation, that is, the computation is similar to the case of implicit plane curves although the formalization and proofs of the results use approaches totally new since the computational techniques and tools in the space are necessarily different to those we have in the plane. Section 5 is devoted to the computation of g-asymptotes for a given parametric space curve. The method presented in this section is totally new. Moreover, it is easily applicable to parametric plane curves and in general, to rational parametrizations of curves in the n-dimensional space. We finish with a section of conclusions (see Section 6) where we summarize the results obtained, we emphasize the new contributions of this paper (compared with those presented in [1,3]), and we propose topics for further study.

2. Notation and terminology

In this section, we present some notions and terminology that will be used throughout the paper. In particular, we introduce some previous results concerning local parametrizations and Puiseux series (see Section 5.2 in [26], Section 2 in [1,27], Section 2.5 in [28,29] and Section 2 in Chapter 4 in [30]). In addition, we generalize the concept of *infinity branch* introduced in [1] (see Section 3) for a given algebraic plane curve. Important results and tools derived from this notion will be presented in the subsequent sections.

We denote by $\mathbb{C}[[t]]$ the domain of *formal power series* in the indeterminate t with coefficients in the field $\mathbb{C}$, i.e. the set of all sums of the form $\sum_{i=0}^{\infty} a_i t^i$, $a_i \in \mathbb{C}$. The quotient field of $\mathbb{C}[[t]]$ is called the field of *formal Laurent series*, and it is denoted by $\mathbb{C}((t))$. It is well known that every non-zero formal Laurent series $A \in \mathbb{C}((t))$ can be written in the form $A(t) = t^k \cdot (a_0 + a_1 t + a_2 t^2 + \cdots)$, where $a_0 \neq 0$ and $k \in \mathbb{Z}$. In addition, the field $\mathbb{C} \ll t \gg := \bigcup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$ is called the field of *formal Puiseux series*. Note that Puiseux series are power series of the form

$$\varphi(t) = m + a_1 t^{N_1/N} + a_2 t^{N_2/N} + a_3 t^{N_3/N} + \cdots \in \mathbb{C} \ll t \gg, \quad a_i \neq 0, \ \forall i \in \mathbb{N},$$

where $N, N_i \in \mathbb{N}$, $i \geq 1$, and $0 < N_1 < N_2 < \cdots$. The natural number N is known as *the ramification index* of the series. We denote it as $\nu(\varphi)$ (see [27]).