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Discrete Legendre spectral projection methods for Fredholm–Hammerstein integral equations

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ABSTRACT

In this paper we discuss the discrete Legendre Galerkin and discrete Legendre collocation methods for Fredholm–Hammerstein integral equations with smooth kernel. Using sufficiently accurate numerical quadrature rule, we obtain optimal convergence rates for both discrete Legendre Galerkin and discrete Legendre collocation solutions in both infinity and L^2-norm. Numerical examples are given to illustrate the theoretical results.

1. Introduction

Let $\mathbb{X} = L^2[-1, 1]$ or $\mathcal{C}[-1, 1]$ and consider the following Hammerstein integral equation

$$x(t) - \int_{-1}^{1} k(t, s)\psi(s, x(s))\,ds = f(t), \quad -1 \le t \le 1, \tag{1.1}$$

where k, f and ψ are known functions and x is the unknown function to be determined. Hammerstein integral equations arise as reformulations of various physical phenomena in different branches of study such as vehicular traffic, biology, economics, etc.

There has been a notable interest in the numerical analysis of solutions of Hammerstein integral equations (1.1) (see [1–7]). The Galerkin, collocation, Petrov–Galerkin, degenerate kernel and Nyström methods are most frequently used projection methods for solving the equations of type (1.1) (see [8,1,9–13]). We are mainly interested in discrete Galerkin and discrete collocation methods in this paper.

Let $\mathbb{X}_n$ be a sequence of finite dimensional approximating subspaces of $\mathbb{X}$ and $\mathcal{P}_n$ be either orthogonal or interpolatory projections from $\mathbb{X}$ onto $\mathbb{X}_n$. Then in Galerkin method ($\mathcal{P}_n$ is orthogonal projection) or in collocation method ($\mathcal{P}_n$ is interpolatory projection), the Hammerstein integral equation (1.1) is approximated by

$$x_n - \mathcal{P}_n\mathcal{K}\psi(x_n) = \mathcal{P}_n f, \tag{1.2}$$

where $\mathcal{K}\psi(x)(t) = \int_{-1}^{1} k(t, s)\psi(s, x(s))\,ds$. Now to apply these projection methods for solving Eq. (1.1), one can use either piecewise polynomial or global polynomials as a basis function of the approximating subspaces.

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In the case of piecewise polynomial based projection methods, we consider $-1 = t_0 < t_1 < \cdots < t_n = 1$, a partition of $[-1, 1]$ and let $h = \max\{t_{i+1} - t_i : 0 \le t_i \le n-1\}$ denote the norm of the partition. We assume that $h \to 0$, as $n \to \infty$. In this case the approximating subspaces $\mathbb{X}_n = S^{\nu}_{r,n}$, the space of all piecewise polynomials of order r (i.e., of degree $\le r-1$) with break points at $t_1, t_2, \ldots, t_{n-1}$ and with ν continuous derivatives, $-1 \le \nu \le r-2$. Here $\nu = 0$ corresponds to the case of continuous piecewise polynomials. If $\nu = -1$, there is no continuity requirements at the break points, in which case we arbitrarily take $x_n \in \mathbb{X}_n$ to be left continuous at $t_1, t_2, \ldots, t_n$ and right continuous at t_0. Note that the dimension of $\mathbb{X}_n$ is

$$N = nr - (n-1)(\nu - 1). \tag{1.3}$$

Under some suitable conditions on the kernel $k(.,.)$ and the right hand side function f of the Eq. (1.1), it is known that the orders of convergence for Galerkin and collocation solutions are $\mathcal{O}(h^r)$ (see [11,14,13]). However, to get better accuracy in piecewise polynomial based projection methods, the number of partition points should be increased. Hence it is evident from (1.3) that in such cases, one has to solve a large system of nonlinear equations, which is computationally very much expensive.

To overcome the computational complexities encountered in the existing piecewise polynomial based projection methods, we apply polynomially-based projection methods to nonlinear Fredholm integral equations (see [15,16]). In [16], the Galerkin and collocation methods were considered for solving Hammerstein integral equation (1.1) using Legendre polynomial basis functions and it was proved that the Legendre Galerkin and Legendre collocation solutions have same orders of convergence, $\mathcal{O}(n^{-r})$ in both infinity and L^2-norm, where n is the highest degree of polynomial employed in the approximation and r is the smoothness of the kernel, the nonlinear function, the right hand side function and the solution. However, the projection methods (1.2) lead to algebraic nonlinear system, in which the coefficients are integrals, appeared due to inner products and integral operator $\mathcal{K}$. These integrals are almost always evaluated numerically. However, in all the above mentioned papers, it was assumed that the integrals appearing in the approximation scheme are evaluated exactly. Thus, in all these methods the effect of error due to numerical integration has been ignored. This motivates to solve these nonlinear systems after replacing the integrals by appropriate numerical integration formula. Replacement of these integrals by numerical quadrature rule gives rise to the discrete projection methods. The effect of quadrature error on the convergence rates of the approximate solution is now considered in these discrete projection methods (see [8,17,9,18–21]). Discrete projection methods for Fredholm nonlinear integral equations with spline bases and their superconvergence results were studied by many authors such as Atkinson and Potra [17], Atkinson and Flores [9], Kumar and Sloan [21] and many others. In [8], Atkinson and Bogomolny have shown that sufficiently accurate quadrature rules can preserve the rates of convergence of the Galerkin method with spline bases.

In this paper, we will apply discrete Galerkin and discrete collocation methods to solve Fredholm–Hammerstein integral equation (1.1) using global polynomials. We choose the approximating subspaces $\mathbb{X}_n$ to be global polynomial subspaces of degree $\le n$, which has dimension $n+1$. The advantage of using global polynomials is that the projection method (1.2) will imply smaller nonlinear systems, something which is highly desirable in practical computations. In particular here, we choose to use Legendre polynomials, which can be generated recursively with ease and possess nice property of orthogonality. Hence from the above discussion it is clear that Legendre polynomials are less expensive computationally compared to piecewise polynomial basis functions. Our purpose in this paper is to obtain similar convergence results in polynomially-based discrete Galerkin and discrete collocation methods for Fredholm–Hammerstein integral equation (1.1) with smooth kernels as in the case of piecewise polynomial based discrete Galerkin and discrete collocation methods. By choosing a numerical quadrature rule appropriately, we show that the discrete Legendre Galerkin and discrete Legendre collocation solutions of the Eq. (1.1) converges with the optimal order $\mathcal{O}(n^{-r})$ in both infinity and L^2-norm, n being the highest degree of the Legendre polynomial employed in the approximation and r is the smoothness of the kernel k, the nonlinear function ψ, the right hand side function f and the solution.

The organization of this paper is as follows. In Section 2, we set up notations and discuss the discrete Legendre Galerkin and discrete Legendre collocation methods for Fredholm–Hammerstein integral equations with smooth kernel. In Section 3, we discuss the existence of the approximate solutions and their convergence rates. In Section 4, we illustrate our results by numerical examples. Throughout this paper, we assume that c is a generic constant.

2. Discrete Legendre Galerkin and discrete Legendre collocation methods: Hammerstein integral equations with smooth kernel

In this section, we describe discrete Galerkin and discrete collocation methods for solving Fredholm–Hammerstein integral equations using global polynomial basis functions.

Let $\mathbb{X} = \mathcal{C}[-1, 1]$ or $L^2[-1, 1]$, with norms $\|\cdot\|_\infty$ and $\|\cdot\|_{L^2}$. For $u \in \mathbb{X}$, we define

$$\|u\|_\infty = \sup_{t\in[-1,1]} |u(t)| \quad \text{and} \quad \|u\|_{L^2} = \left(\int_{-1}^{1} |u(t)|^2 dt\right)^{\frac{1}{2}}.$$

Consider the following Hammerstein integral equation

$$x(t) - \int_{-1}^{1} k(t,s)\psi(s, x(s))\, ds = f(t), \quad -1 \le t \le 1, \tag{2.1}$$