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Generalized quadrature rules of Gaussian type for numerical evaluation of singular integrals[☆]

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ABSTRACT

An efficient method for constructing a class of generalized quadrature formulae of Gaussian type on $(-1, 1)$ for integrands having logarithmic singularities is developed. That kind of singular integrals are very common in the boundary element method. Several special cases for n-point quadratures, which are exact on both of the spaces $\mathcal{P}_{2n-2\ell-1}[-1, 1]$ (the space of algebraic polynomials of degree at most $2n - 2\ell - 1$) and $\mathcal{L}_{2\ell-1}[-1, 1] = \text{span}\{x^k \log |x|\}_{k=0}^{2\ell-1}$ (the logarithmic space), where $1 \le \ell \le n$, are presented. Regarding a direct connection of these $2m$-point quadratures with m-point quadratures of Gaussian type with respect to the weight function $t \mapsto t^{-1/2}$ over $(0, 1)$, the method of construction is significantly simplified. Gaussian quadratures on $(0, 1)$ are exact for integrands of the form $t \mapsto p(t) + q(t) \log t$, where p and q are algebraic polynomials of degree at most $2m - \ell - 1$ and $\ell - 1$ $(1 \le \ell \le 2m)$, respectively. The obtained quadratures can be used in a software implementation of the boundary element method.

1. Introduction and preliminaries

The finite element method (FEM) and the boundary element method (BEM) are very popular in many computational applications in engineering, e.g., in fracture mechanics, damage mechanics, heat transfer problems, fluid flow problems, electromagnetic problems, etc. A boundary value problem, described by a differential equation and the corresponding boundary condition, can be modeled numerically by the BEM, using a discretization which leads to a system of linear algebraic equations, so that this system approximates the solution of the original problem. The accuracy of this approximation and the efficiency of the BEM depend on the boundary discretization technique (i.e., the type of employed elements), as well as the quadrature method used for the integration of the kernel functions over the elements. A systematic derivation of various integral equation formulations for solution of boundary value problems in potential theory with continuously variable material coefficients has been recently given in [1].

A general classification of boundary elements, as well as the theory and applications of BEM, can be found in the nice books of Katsikadelis [2], Beer, Smith and Duenser [3], and Sauter and Schwab [4] (see also [5] for FEM). In implementation of the Galerkin method for boundary integral equations as a very important task is an approximation of the coefficients of the system matrix and the right-hand side. Such kind of problems is considered in [4, Chapter 5], including numerical integration in one dimension and in several dimensions using the so-called tensor-Gauss quadrature. In general, quadrature formulas

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play a very important role in numerical implementation of the BEM, especially for higher order elements (see [2, Chapters 4 & 5] and [3, Chapter 6]). For calculating integrals of the corresponding influence coefficients (for off-diagonal elements and diagonal elements), quadratures of Gaussian type are very appropriate. For sufficiently smooth functions on a finite interval $[a, b]$ a linear transformation to the standard interval $[-1, 1]$ can be used and then an application of Gauss–Legendre quadrature formula provides numerical integration with a satisfactory accuracy. However, for integrals with a logarithmic singularity the convergence of the corresponding quadrature process is very slow, so that certain weighted quadratures of Gaussian type are recommended, e.g.,

$$\int_0^1 f(x)\log x\,\mathrm{d}x \approx \sum_{k=1}^{n} w_k^L f(x_k^L). \tag{1.1}$$

The parameters (nodes x_k^L and weights coefficients w_k^L, $k = 1, \ldots, n$) of the last formula can be found in many books (see, for example, [2, Appendix B, pp. 289–308], where the quadrature parameters are listed for $n = 2(1)8$). In such kind of Gaussian quadratures, the weight functions include these "difficult parts (with singularities)" of the integrand.

Integrals with nearby and strong singularities was recently considered by Tsamasphyros and Theotokoglou [6].

It would be very useful to develop some kind of quadrature rules suited for integrands with and without logarithmic singularities. In other words, such universal (direct) quadrature formulae need to be able to calculate integrals with a sufficient accuracy, regardless of whether their integrands contain a logarithmic singularity, or they do not. This would avoid the separation into singular and non-singular parts in integrands, as well as an additional integration of such a singular part using some special logarithmically weighted quadrature formula like (1.1).

An approach for constructing universal quadrature formulae which integrate both kind of functions, smooth and ones with a logarithmic singularity, was considered by Nahlik and Białecki [7]. Namely, supposing that the integrand behaves like a logarithm in the vicinity of $x = 0$, i.e., when

$$f(x) = C_1 \log|x| + C_2 + C_3 x + C_4 x^2 + \cdots \quad (C_k \text{ are constants}),$$

they considered quadrature formulas of the form

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = \sum_{k=1}^{n} A_k f(x_k) + R_n(f). \tag{1.2}$$

Using the Gram–Schmidt orthogonalization procedure with respect to the inner product $(f, g) = \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x$, they started from the system of functions

$$\left\{\log|x|,\ 1,\ x,\ x^2,\ x^3,\ x^4,\ x^5,\ x^6\right\}$$

and obtained an orthogonal system of functions $\{g_\nu\}_{\nu=1}^{8}$, where only the even elements of this system are of interest in their construction. In order to construct $2m$-point quadratures, for $m = 2, 3,$ and 4, they choose the zeros of

$$g_4(x) = -\frac{2}{9}\log|x| - \frac{5}{9} + x^2, \qquad g_6(x) = \frac{12}{175}\log|x| + \frac{37}{175} - \frac{36}{35}x^2 + x^4,$$

and

$$g_8(x) = -\frac{10}{539}\log|x| - \frac{69}{1078} + \frac{45}{77}x^2 - \frac{225}{154}x^4 + x^6,$$

respectively, as the nodes in (1.2). These zeros are real, distinct and are symmetrically distributed in $(-1, 1)$. The weight coefficients are determined from the corresponding system of linear equations. Quadratures obtained in this way are not of Gaussian type.

Another approach for weakly singular logarithmic integrals, which appear in two-dimensional BEM problems, was considered by Smith [8]. He discussed some direct Gaussian rules for logarithmic singularities on isoparametric (quadratic and cubic) elements. These direct quadratures are superior to conventional ones, their implementation reduces the size of the computer program, and they can be used also for constant and linear elements (see [8]).

In this paper we give direct Gaussian rules for logarithmic singularities which are more general than ones in [7,8]. Namely, we consider symmetric quadrature rules (1.2) of Gaussian type, with an even number of nodes, $n = 2m$. The parameters of such formulas, nodes and weights, satisfy the relations

$$x_{n+k} = -x_k, \qquad A_{n+k} = A_k, \quad k = 1, \ldots, m. \tag{1.3}$$

Therefore, without loss of generality, we suppose that the formula (1.2) can be represented in the form

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = \sum_{k=1}^{m} A_k\big(f(x_k) + f(-x_k)\big) + R_n(f), \tag{1.4}$$

with $0 < x_1 < x_2 < \cdots < x_m < 1$. The corresponding remainder term in (1.2), i.e., (1.2), is denoted by $R_n(f)$. Evidently, $R_n(f) = 0$ for all odd functions.

In order to have Gaussian quadrature formulas for functions with a logarithmic singularity at the origin we need the following concept.