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Numerical approximation of solution derivatives in the case of singularly perturbed time dependent reaction–diffusion problems

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ABSTRACT

Numerical approximations to the solution of a linear singularly perturbed parabolic problem are generated using a classical finite difference operator on a piecewise-uniform Shishkin mesh. First order convergence of these numerical approximations in an appropriately weighted C^1-norm is established. Numerical results are given to illustrate the theoretical error bounds.

1. Introduction

Singularly perturbed differential equations are typically characterized by the presence of a small positive parameter (denoted here by ε) multiplying some or all of the highest derivatives present in the differential equation. These equations arise in numerous applications [1,2], once the physical variables have been reformulated into dimensionless variables. Fluid dynamics, chemical kinetics, combustion and control theory are sample areas from science where singularly perturbed problems naturally appear.

It has been well established [3] that classical numerical approaches to solving these problems have many deficiencies, especially if one is interested in pointwise global accuracy or in estimating physically meaningful quantities such as the flux.

Nodal accuracy is a necessary, but not a sufficient condition for global accuracy. In addition, the level of nodal accuracy may depend crucially on the location of the mesh points. In the context of singularly perturbed problems, high levels of accuracy can be obtained if there are no mesh points located within the thin layer regions. This effect is caused by the fact that within the layer regions the magnitude of the first derivative of the solution can be significantly larger compared to the size of the solution derivatives outside the layers. Moreover, the size of any layer region shrinks to zero as the singular perturbation parameter tends to zero. Hence, unless the mesh is layer-adapted, for sufficiently small values of the perturbation parameter there will be no mesh points within the layer. For this reason, we are interested in measuring the pointwise accuracy of numerical approximations at all points within the domain of the continuous problem, both within and outside the layers. Hence, our interest is not only in parameter-uniform nodal convergence, but in parameter-uniform global convergence [3].

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Since the seminal papers of Bakhvalov [4] and Il'in [5], parameter-uniform numerical methods have been developed for several classes of singularly perturbed problems. These parameter-uniform methods generate numerical approximations, whose pointwise accuracy is retained independently of the singular perturbation parameter. In [6] Shishkin established that a uniform mesh is not an adequate discretization of the domain, if one wants to design a parameter-uniform numerical method for a class of singularly perturbed time dependent reaction–diffusion problems. For the approximations to have a pointwise accuracy independent of the perturbation parameter, a non-uniform (layer-adapted) mesh is a necessary component for any parameter-uniform numerical method, when dealing with a class of problems of the form

$$-\varepsilon u_{xx} + bu + u_t = f, \quad (x,t) \in G := (0,1) \times (0,1], \tag{1a}$$

$$u = g, \quad (x,t) \in \bar{G} \setminus G. \tag{1b}$$

Moreover, in [6] Shishkin created his now well-known piecewise-uniform mesh to generate such parameter-uniform approximations to the solutions of problems from this class of time dependent reaction–diffusion problems. Shishkin [7] and others have subsequently continued to show that piecewise-uniform meshes can be used to create parameter-uniform numerical methods for an extensive class of singularly perturbed problems. In this paper, we return to the same class of singularly perturbed reaction–diffusion problems as in [6], to examine other attributes of the Shishkin mesh.

For the problem class (1), it was established in [8] that for a numerical solution U^N generated using a standard finite difference operator and an appropriate piecewise-uniform Shishkin mesh, one has a global error bound of the form

$$\|u - \bar{U}\|_G \le C(N^{-1}\ln N)^2 + CM^{-1}, \tag{2}$$

where $\bar{U}$ denotes a piecewise bilinear interpolant over the domain $\bar{G}$ of the discrete solution $U^{N,M}$, and N, M denote the number of mesh points used in the space and time directions, respectively; and we define the global pointwise norm

$$\|u\|_G := \operatorname*{ess\,sup}_{(x,t)\in G} |u(x,t)|.$$

Throughout this paper, C denotes a generic constant that is independent of the singular perturbation parameter ε and of all discretization parameters.

An additional feature of Shishkin meshes is that accurate approximations to the scaled flux can be easily generated. Numerical evidence for this was given in [3,8]. In this paper a proof of the convergence of the scaled discrete derivatives is given. As the derivatives of the solution are only large in the boundary layers, it is only necessary to scale the derivatives in the boundary layer regions. For this reason, in this paper, we will derive an error estimate in the following weighted C^1-norm:

$$\|v\|_{1,\kappa,G} := \|\kappa(x) v_x\|_G + \|v_t\|_G + \|v\|_G, \tag{3}$$

with

$$\kappa(x) := \begin{cases} \sqrt{\varepsilon}, & \text{if } x \in \left(0, \sqrt{\frac{\varepsilon}{\beta}}\ln\frac{1}{\varepsilon}\right) \cup \left(1 - \sqrt{\frac{\varepsilon}{\beta}}\ln\frac{1}{\varepsilon}, 1\right), \\ 1, & \text{otherwise.} \end{cases}$$

Shishkin et al. [9], [7, pp. 353–354] have highlighted the difficulties in selecting a weighted C^1-norm to appropriately measure the relative error in numerical approximations to the solutions of singularly perturbed problems. In particular, observe that the weight $\kappa(x)$ in the above norm is excessively small in the region where $C\sqrt{\varepsilon} \ll \alpha|x-p| \le \sqrt{\frac{\varepsilon}{\beta}}\ln(1/\varepsilon),\ p = 0, 1$. Nevertheless, outside this subregion, the norm $\|\cdot\|_{1,\kappa,G}$ is an appropriate norm to use in the context of problem (1).

This paper is structured as follows: in Section 2 the continuous problem is stated and parameter-explicit bounds on the derivatives of the solution are given. In Section 3, the numerical method is described and in Section 4 scaled nodal approximations of the space and time derivatives, respectively, are deduced. The main result of the paper, which establishes the error estimate

$$\|\bar{U} - u\|_{1,\kappa,G} \le CN^{-1}\ln N,$$

is given in Section 5. Some numerical results are presented in the final section of the paper.

2. Continuous problem

Consider the following class of singularly perturbed parabolic problems:

$$L_\varepsilon u := -\varepsilon u_{xx} + b(x,t)u + c(x,t)u_t = f(x,t), \quad \text{in } G := \Omega \times (0,T], \tag{4a}$$

$$u = 0, \quad \text{on } \Gamma_B \cup \Gamma_L \cup \Gamma_R,\ 0 < \varepsilon \le 1; \tag{4b}$$

$$b(x,t) \ge \|c_t\|_G + \beta > 0, \qquad c(x,t) \ge c_0 > 0, \tag{4c}$$

where $0 < \varepsilon \le 1$, $\Omega := (0,1)$, $\Gamma_B := \{(x,0) \mid 0 \le x \le 1\}$, $\Gamma_L := \{(0,t) \mid 0 \le t \le T\}$, $\Gamma_R := \{(1,t) \mid 0 \le t \le T\}$ and $\Gamma := \bar{G} \setminus G$. Since the problem is linear, there is no loss in generality in assuming zero boundary/initial conditions. The