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Multidimensional fixed point theorems under (ψ, φ)-contractive conditions in partially ordered complete metric spaces

A. Roldán [a], J. Martínez-Moreno [a,*], C. Roldán [b], Y.J. Cho [c,d]

[a] Department of Mathematics, University of Jaén, Spain
[b] Department of Statistics and Operations Research, University of Granada, Spain
[c] Department of Mathematics Education and RINS, Gyeongsang National University, Republic of Korea
[d] Department of Mathematics, King Ábdulaziz University, Jeddah, Saudi Arabia

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ABSTRACT

In this paper, we study the existence and uniqueness of coincidence points for nonlinear mappings of any number of arguments under a weak (ψ, φ)-contractive condition. Our results generalize, extend and unify several classical and very recent related results in the literature (see Aydi et al. (2011), Berinde (2010), Gnana-Bhaskar and Lakshmikantham (2006), Berzig and Samet (2012), Borcut and Berinde (2012), Choudhury et al. (2011), Karapınar and Luong (2012), Lakshmikantham and Ćirić (2009), Luong and Thuan (2011), Roldán (2012)).

1. Introduction

The concept of *coupled fixed point* for nonlinear operators was introduced and studied by Opoitsev (see [1–3]) and then, in 1987, by Guo and Lakshmikantham (see [4]) in connection with coupled quasisolutions of an initial value problem for ordinary differential equations. In a recent paper, Gnana-Bhaskar and Lakshmikantham [5] introduced the concept of the *mixed monotone property* for contractive mappings of the form $F : X \times X \to X$, where X is a partially ordered metric space, and then established some coupled fixed point theorems. Since then, many results on coupled fixed point theory in different contexts have been published (see [6–12]). Later, Berinde and Borcut [13] introduced the concept of the *tripled fixed point* and proved tripled fixed point theorems using the mixed monotone mappings (see also [14–16]).

Very recently, Roldán et al. [17] proposed the notion of *coincidence point* between mappings in any number of variables and showed some existence and uniqueness theorems that extend the mentioned previous results for this kind of nonlinear mappings, not necessarily permuted or ordered, in the framework of partially ordered complete metric spaces, using a weaker contraction condition, that also generalizes other work by Berzig and Samet [18].

In this paper, our main aim is to study a weaker contractive condition for nonlinear mappings of any number of arguments. This condition can be particularized in a variety of forms that lets us to extend the above mentioned results and other recent ones in this field (see [5,10,13,15,17–22]).

* Correspondence to: University of Jaén, Campus las Lagunillas s/n, 23071, Jaén, Spain. Tel.: +34 953212931; fax: +34 683750070.
E-mail address: jmmoreno@ujaen.es (J. Martínez-Moreno).

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2. Preliminaries

Preliminaries and notation about coincidence points can also be found in [17]. Let n be a positive integer. Henceforth, X denotes a nonempty set and X^n denotes the product space $X \times X \times \cdots \times X$. Throughout this manuscript, m and k denote non-negative integers and $i, j, s \in \{1, 2, \ldots, n\}$. Unless otherwise stated, "for all m" means "for all $m \geq 0$" and "for all i" means "for all $i \in \{1, 2, \ldots, n\}$".

A *metric* d on X is a mapping $d : X \times X \to \mathbb{R}$ satisfying: for all $x, y, z \in X$,

(a) $d(x, y) = 0$ if and only if $x = y$; (b) $d(x, y) \leq d(z, x) + d(z, y)$.

From these properties, we can easily deduce that $d(x, y) \geq 0$ and $d(y, x) = d(x, y)$ for all $x, y \in X$. The last requirement is called the *triangle inequality*. If d is a metric on X, we say that (X, d) is a *metric space*.

Definition 1 (*[23]*). A triple $(X, d, \leq)$ is called an *ordered metric space* if (X, d) is a metric space and $(X, \leq)$ is a partially ordered set.

In the sequel, let $F : X^n \to X$ and $g : X \to X$ be two mappings.

Definition 2 (*[5]*). An ordered metric space $(X, d, \leq)$ is said to have the *sequential g-monotone property* if it verifies the following properties:

(a) If $\{x_m\}$ is a non-decreasing sequence and $\lim_{m\to\infty} x_m = x$, then $gx_m \leq gx$ for all m.
(b) If $\{y_m\}$ is a non-increasing sequence and $\lim_{m\to\infty} y_m = y$, then $gy_m \geq gy$ for all m.

If g is the identity mapping, then X is said to have the *sequential monotone property*.

Henceforth, we fix a partition $\{A, B\}$ of $\Lambda_n = \{1, 2, \ldots, n\}$, that is, $A \cup B = \Lambda_n$ and $A \cap B = \varnothing$. Then we denote:

$$\Omega_{A,B} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq A \text{ and } \sigma(B) \subseteq B\}$$

and

$$\Omega'_{A,B} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq B \text{ and } \sigma(B) \subseteq A\}.$$

If $(X, \leq)$ is a partially ordered space, $x, y \in X$ and $i \in \Lambda_n$, then we use the following notation:

$$x \leq_i y \iff \begin{cases} x \leq y & \text{if } i \in A, \\ x \geq y & \text{if } i \in B. \end{cases}$$

Definition 3 (*[17]*). Let $(X, \leq)$ be a partially ordered space. We say that F has the *mixed g-monotone property* with respect to the partition $\{A, B\}$ if F is g-monotone non-decreasing in arguments of A and g-monotone non-increasing in arguments of B, i.e., for all $x_1, x_2, \ldots, x_n, y, z \in X$ and i,

$$gy \leq gz \implies F(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \leq_i F(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n).$$

Henceforth, let $\sigma_1, \sigma_2, \ldots, \sigma_n, \tau : \Lambda_n \to \Lambda_n$ be $n + 1$ mappings from Λ_n into itself and let Φ be the $(n + 1)$-tuple $(\sigma_1, \sigma_2, \ldots, \sigma_n, \tau)$.

Definition 4 (*[17]*). A point $(x_1, x_2, \ldots, x_n) \in X^n$ is called a *Φ-coincidence point* of the mappings F and g if, for all i,

$$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}) = gx_{\tau(i)}. \tag{1}$$

If g is the identity mapping on X, then $(x_1, x_2, \ldots, x_n) \in X^n$ is called a *Φ-fixed point* of the mapping F.

Remark 5. If F and g are commuting and $(x_1, x_2, \ldots, x_n) \in X^n$ is a Φ-coincidence point of F and g, then $(gx_1, gx_2, \ldots, gx_n)$ also is a Φ-coincidence point of F and g.

The commutative condition between F and g is very restrictive. When $n = 2$, in [24], the authors introduced a weaker condition, which we will extend to any number of variables.

Definition 6 (*[24,25]*). Let $(X, d, \leq)$ be an ordered metric space. The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be *O-compatible* if

$$\lim_{m\to\infty} d(gF(x_m, y_m), F(gx_m, gy_m)) = 0, \qquad \lim_{m\to\infty} d(gF(y_m, x_m), F(gy_m, gx_m)) = 0,$$

where $\{x_m\}$ and $\{y_m\}$ are the sequences in X such that $\{gx_m\}$, $\{gy_m\}$ are monotone and

$$\lim_{m\to\infty} F(x_m, y_m) = \lim_{m\to\infty} gx_m = x, \qquad \lim_{m\to\infty} F(y_m, x_m) = \lim_{m\to\infty} gy_m = y$$

for all $x, y \in X$.