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Semilocal convergence by using recurrence relations for a fifth-order method in Banach spaces[☆]

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ABSTRACT

In this paper, a semilocal convergence result in Banach spaces of an efficient fifth-order method is analyzed. Recurrence relations are used in order to prove this convergence, and some a priori error bounds are found. This scheme is finally used to estimate the solution of an integral equation and so, the theoretical results are numerically checked. We use this example to show the better efficiency of the current method compared with other existing ones, including Newton's scheme.

1. Introduction

Newton's method and its variants are used to solve nonlinear equations of the form $F(x) = 0$. This equation can represent differential equations, integral equations or a system of nonlinear equations. The convergence of Newton's method in Banach spaces was established by Kantorovich in [1]. The convergence of the sequence obtained by the iterative expression is derived from the convergence of majorizing sequences. This technique has been used by many authors in order to establish the order of convergence of the variants of Newton's methods (see, for example, [2,3]).

Rall in [4] suggested a different approach for the convergence of these methods, based on recurrence relations. Amat, Hernández and Romero [5,6], Ezquerro and Hernández [7], Gutiérrez and Hernández [8,9], Parida and Gupta in [10] and Candela and Marquina [11,12] used this idea to prove the semilocal convergence for several methods of different orders.

In this paper, we analyze the semilocal convergence of a fifth-order method M5 considered in [13] for solving systems of nonlinear equations. In order to get this aim, we use the technique of recurrence relations, that consists of generating a sequence of positive real numbers that guarantees the convergence of the iterative scheme in Banach spaces, providing a suitable convergence domain. This technique allows us to establish weak semilocal convergence conditions for an iterative method with fifth-order of convergence. Even more, we get a result of semilocal convergence under the same conditions of Kantorovich Theorem for Newton's method, which has quadratic convergence. This allows us to apply the fifth-order convergence method for solving nonlinear equations $F(x) = 0$ under the same conditions that assures us the convergence of Newton's method.

Another important aspect of this work is the comparative study of the efficiency of the proposed scheme with the one of other known high-order methods, such as Jarratt's method (see [14]) and the one introduced by Wang et al. in [15], by using

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the classical efficiency index defined by Ostrowski in [16] and the computational efficiency index described by Traub in [17]. In addition, we include in our comparative study of the efficiency the most used iterative process, the Newton method. Noting that the proposed iterative process M5 is also more efficient than Newton's method when trying to approximate a solution of a system with more than two equations.

Finally, we make some test on integral equations in order to check the theoretical results. Noting that the proposed method M5 is more efficient for the approximation of a solution.

The rest of the paper is organized as follows: in Section 2 we describe the recurrence relations and the properties needed to prove the semilocal convergence of method M5 in Section 3. In Section 4 the comparative analysis of the efficiency is made. Finally, in Section 5 an application on integral equation of mixed Hammerstein type is illustrated.

2. Recurrence relations

Let X, Y be Banach spaces and $F : \Omega \subseteq X \to Y$ be a nonlinear twice Fréchet differentiable operator in an open convex domain Ω. The fifth-order method M5, which semilocal convergence we are going to study can be found in [13] and its iterative expression is:

$$\begin{cases} y_n = x_n - \Gamma_n F(x_n), \\ z_n = y_n - 5\Gamma_n F(y_n), \\ x_{n+1} = z_n - \dfrac{1}{5}\Gamma_n\left(-16F(y_n) + F(z_n)\right), \end{cases} \tag{1}$$

where $\Gamma_n = \left[F'(x_n)\right]^{-1}$, for $n \in \mathbb{N}$.

Let us assume that the inverse of F' at x_0, $F'(x_0)^{-1} = \Gamma_0 \in \mathcal{L}(Y, X)$ exists at some $x_0 \in \Omega$, where $\mathcal{L}(Y, X)$ is the set of bounded linear operators from Y into X.

In the following we will assume that $y_0, z_0 \in \Omega$ and

(i) $\|\Gamma_0\| \le \beta$,
(ii) $\|\Gamma_0 F(x_0)\| \le \eta$,
(iii) $\|F'(x) - F'(y)\| \le K\|x - y\|,\ x, y \in \Omega$,

in order to obtain the recurrence relations that satisfy the steps that appear in the iterative process (1).

Notice that these are the classical Kantorovich's conditions [1] for the semilocal convergence of Newton's method.

Let us also denote by $a_0 = K\beta\eta$ and define the sequence $a_{n+1} = a_n f(a_n)^2 g(a_n)$, where

$$f(x) = \frac{1}{1 - x(h(x) + 1)}, \tag{2}$$

$$g(x) = \frac{1}{2}x + (x + 1)h(x) + \frac{1}{2}xh(x)^2 \tag{3}$$

and

$$h(x) = \frac{1}{2}x + \frac{1}{2}x^2 + \frac{5}{8}x^3. \tag{4}$$

To study the convergence of $\{x_n\}$ defined by (1) to a solution of $F(x) = 0$ in a Banach space, we have to prove that $\{x_n\}$ is a Cauchy sequence. To do this, we need to analyze some properties of sequence $\{a_n\}$ and, previously, of the real functions described in (2)–(4), respectively.

Lemma 1. *Let $f(x)$, $g(x)$ and $h(x)$ be the real functions described in (2)–(4). Then,*

(i) *f is increasing and $f(x) > 1$ for $x \in (0, 0.6)$,*
(ii) *h and g are increasing for $x \in (0, 0.6)$.*

Lemma 2. *Let $f(x)$ and $g(x)$ as before and $a_0 \in (0, 0.2931\ldots)$. Then,*

(i) $f(a_0)^2 g(a_0) < 1$,
(ii) $f(a_0) g(a_0) < 1$,
(iii) *the sequence $\{a_n\}$ is decreasing and $a_n < 0.2931\ldots$, for $n \ge 0$.*

Proof. From the definition of functions f and g (i) follows trivially. From (i) and $f(a_0) > 1$, we obtain (ii). We are going to prove (iii) by induction on $n \ge 0$. Firstly, from (i) and the definition of a_1, we have that $a_1 < a_0$. Now, it is supposed that $a_k < a_{k-1}$, for $k \le n$. Then,

$$a_{n+1} = a_n f(a_n)^2 g(a_n) < a_{n-1} f(a_n)^2 g(a_n) < a_{n-1} f(a_{n-1})^2 g(a_{n-1}) = a_n,$$

as f and g are increasing and $f(x) > 1$.

Finally, for all $n \ge 0$, $a_n < 0.2931\ldots$, since $\{a_n\}$ is a decreasing sequence and $a_0 < 0.2931\ldots$. □