Journal of Computational and Applied Mathematics 273 (2015) 214–224

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

An upper bound of the Bezout number for piecewise algebraic curves[☆]

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ARTICLE INFO

Article history:
Received 2 July 2013
Received in revised form 25 March 2014

Keywords:
Bezout number
Piecewise algebraic curves
Homogeneous trigonometric periodic splines
Special partitions

ABSTRACT

A piecewise algebraic curve is a curve determined by the zero set of a bivariate spline function. Based on the discussion of the number of the zeros of homogeneous trigonometric splines with different smoothness and the common points of two piecewise algebraic curves over a star partition, a better upper bound of Bezout number of two piecewise algebraic curves over arbitrary triangulation is found. Moreover, upper bounds of the Bezout number $BN(m, r; n, r; \Delta)$ for piecewise algebraic curves over several special partitions such as rectangular partition, type-1 triangulation and type-2 triangulation are obtained.

1. Introduction

Let Ω be a connected bounded region in $\mathbb{R}^2$. Using a finite number of irreducible algebraic curves to carry out the partition Δ in domain Ω, the domain Ω is divided into a finite number of sub-domains (generally convex) $D_1, D_2, \ldots, D_N$ by the partition Δ; each of such sub-domains is called a "cell". These line segments that form the boundary of each cell are called the "mesh segments"; intersection points of the mesh segments are called the "mesh points". Also, we write $\Delta = \cup_{i=1}^{N} D_i$. If we set up a partition Δ of Ω as follows: all the mesh segments are straight lines cross-cut domain Ω. This kind of partition is called a cross-cut partition. It includes tiling, parallel partition and so on. Another important partition is the classical triangulation. It includes many partitions (type-1 partition and type-2 partition) as its special cases. From now on, we only focus on the arbitrary triangulation or some special cross-cut partitions.

For a vertex, star $st(v)$ means the union of all cells in Δ sharing v as a common vertex, and its degree $d(v)$ is defined to be the number of the edges sharing v as a common endpoints. If $d(v)$ is odd, we call v an odd vertex.

Let $\mathbb{P}_k[x, y]$ denote the set of bivariate polynomials in $\mathbb{R}[x, y]$ with real coefficients and total degree $\leqslant k$ and denoted by $C^\mu(\Delta)$ the set of C^μ functions s on Ω such that for every $D_i \in \Delta$. Assume the restriction $s|_{D_i}$ is a polynomial function which belongs to $\mathbb{P}_k[x, y]$ and let

$$\mathbb{P}_k(\Delta) := \{s(x, y) \mid s^{[i]}(x, y) = s(x, y)|_{D_i} \in \mathbb{P}_k[x, y],\ i = 1, 2, \ldots, T\}$$

be the set of piecewise polynomials defined on Δ with total degree $\leqslant k$.

☆ This work was supported by the National Natural Science Foundation of China (Nos. 11101366, 11026086, 11271328 and 61272307), the Zhejiang Provincial Natural Science Foundation of China (Nos. LY14A010001 and LQ13A010004).

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http://dx.doi.org/10.1016/j.cam.2014.06.015
0377-0427/

The bivariate spline space with degree k and smoothness μ over Ω with respect to Δ is defined as follows

$$S_k^{\mu}(\Delta) := \left\{ s(x,y) \mid s(x,y) \in C^{\mu}(\Delta) \cap P_k(\Delta) \right\}.$$

The zero set

$$z(s) := \left\{ (x,y) \mid s(x,y) = 0,\ s(x,y) \in S_k^{\mu}(\Delta) \right\}$$

is called a piecewise algebraic curve. Obviously, the piecewise algebraic curve is a generalization of the classic algebraic curve [1].

We notice the link between the partitioning of the domain Ω and the correspondence to pieces of algebraic curves $\mathcal{C}_i = \{(x,y) \mid s(x,y)|_{D_i} = 0\}$. Specifically, each D_i corresponds to the algebraic curve $\mathcal{C}_i$ and $\cup_{i=1}^{N} D_i$ forms the domain of the piecewise algebraic curve $Z(s)$.

Piecewise algebraic curve is originally introduced by Prof. Wang in the study of bivariate spline interpolation. He pointed out that the given interpolation knots are properly posed if and only if they are not lying on a non-zero piecewise algebraic curve. Hence, it is necessary and important to deal with the piecewise algebraic curve in order to solve bivariate spline interpolation and related problems [2–4]. Piecewise algebraic curve is a new and important topic of computational geometry and algebraic geometry, and it is of important theoretical and practical significance in many fields.

Bezout's theorem is a statement in algebraic geometry concerning the number of common points, or intersection points, of two plane algebraic curves. The theorem claims that in the complex projective plane, if two curves have no common component, the number of isolated solutions counted with multiplicities equals the product of the degrees of the curves. A simpler special case is when one does not care about multiplicities and X and Y are two algebraic curves in the Euclidean plane whose implicit equations are polynomials of degrees m and n without any non-constant common factor; then the number of intersection points does not exceed mn. The number mn is called Bezout number. For examples, two distinct non-parallel lines always meet in exactly one point and two parallel lines intersect at a unique point that lies at infinity. In complex coordinates and including the points on the infinite line in the projective plane, two conic sections generally intersect in four points.

For bivariate spline spaces, we denote $BN := BN(m, r; n, t; \Delta)$ the Bezout number for $S_m^r(\Delta)$ and $S_n^t(\Delta)$. It means that if two piecewise algebraic curves $f(x,y) = 0$ and $g(x,y) = 0$, where $f(x,y) \in S_m^r(\Delta),\ g(x,y) \in S_n^t(\Delta)$ must have infinitely many intersection points provided that they have more than BN intersection points. Needless to say, it is crucial for us to obtain $BN(m, r; n, t; \Delta)$. It mentioned that by utilizing the smoothness it is possible to reduce the Bezout number. If $r \leqslant t$, then we have

$$BN(m, t; n, t; \Delta) \leqslant BN(m, r; n, t; \Delta) \leqslant BN(m, r; n, r; \Delta).$$

However, we remark that it is very hard to obtain exact $BN(m, r; n, t; \Delta)$. On the one hand, piecewise algebraic curve itself is difficult; On the other hand, the exact Bezout number $BN(m, r; n, t; \Delta)$ is difficult to obtain since it not only relies on the degrees m, n and the smoothness orders r, t, but also depends heavily on the geometric characteristics of Δ. Hence, we can only get an upper bound for $BN(m, r; n, t; \Delta)$. Several researches (Prof. Wang, Luo, Zhao, Xu et al.) have make several important and excellent works about the Bezout number for piecewise algebraic curves.

Denoted by Δ_k the partition of Ω consisting of k parallel lines. The first work is Luo's Ph.D. thesis, which provides the sharply upper bound of Bezout number for piecewise algebraic curves on the partition by the parallel lines.

Lemma 1.1 (*[5]*).

$$BN(m, r; n, t; \Delta_k) \leqslant (k+1)mn - \min\{r, t\}k.$$

Take $BN(2, 1; 2, 1; \Delta_1)$ for example, we know that for arbitrary two piecewise algebraic curves $f(x,y) = 0$ and $g(x,y) = 0$ has at most 7 intersection points over Δ_1, where $f(x,y),\ g(x,y) \in S_2^1(\Delta_1)$ and Δ_1 denotes the partition consisting of one line.

The triangulation Δ is called non-obtuse triangulation if none of the triangles in the Δ is obtuse-angled. Zhao gave the sharply upper bound of Bezout number for piecewise algebraic curves over a non-obtuse star partition by using the polar coordinates.

Lemma 1.2 (*[6]*). *Let Δ be a non-obtuse triangulation. If v is an interior vertex of Δ. Then*

$$BN(m, 1; n, 1; st(v)) \leqslant d(v)mn - (d(v) - 1).$$

Lemma 1.3 (*[6]*). *Let Δ be a non-obtuse triangulation. If v is an boundary vertex of Δ. Then*

$$BN(m, 1; n, 1; st(v)) \leqslant (d(v) - 1)mn - (d(v) - 2).$$

In 1999, Shi et al. gave a sharply upper bound of the Bezout number for two continuous piecewise algebraic curves on arbitrary triangulation.